Efficient scheduled stabilizing model predictive control for constrained nonlinear systems

Zhaoyang Wan and Mayuresh V. Kothare*

Department of Chemical Engineering, Chemical Process Modeling and Control Research Center, Lehigh University, 111 Research Drive, Bethlehem, PA 18015, U.S.A.

SUMMARY
We present a computationally efficient scheduled model predictive control (MPC) algorithm for constrained nonlinear systems with large operating regions. We design a set of local predictive controllers with estimates of their regions of stability covering the desired operating region, and implement them as a single scheduled MPC which on-line switches between the set of local controllers and achieves nonlinear transitions with guaranteed stability. This algorithm is computationally efficient and provides a general framework for the scheduled MPC design. The algorithm is illustrated with two examples. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: model predictive control; constrained nonlinear systems; gain scheduling; invariant ellipsoid; linear matrix inequalities

1. INTRODUCTION
Most practical control systems with large operating regions must deal with both nonlinearity and constraints. Nonlinear Model Predictive Control (NMPC) handles constraints on the manipulated and controlled variables explicitly by solving a finite-horizon optimization problem at each sampling time. In general, stability of NMPC is enforced by means of suitable penalties and constraints on the state at the end of the finite optimization horizon [1–6]. Since the feasibility of NMPC is determined through on-line optimization, as a drawback, little attempt has been made to estimate explicitly the region of stability for most NMPC algorithms. Moreover, the region of stability for most NMPC formulations can only be characterized qualitatively as ‘at least as large as that of the auxiliary linear controller’ based on the linearized dynamics [4]. Furthermore, since NMPC generally requires the computation of an optimal or suboptimal solution of a nonlinear non-convex optimization problem at each sampling time, the
enlargement of its region of stability at the cost of an increase of the finite horizon further intensifies the on-line computation.

Gain scheduling is an attractive practical approach to control systems with large operating regions. In Reference [7], a novel gain scheduling procedure was introduced which provides stability guarantees between the scheduling points. A related yet more general and theoretical hierarchical switching control was proposed in Reference [8]. Both approaches require an estimate of the region of stability for the local controller designed at each scheduling point. And by designing a set of local controllers with their estimated regions of stability overlapping each other, supervisory scheduling of the local controllers can move the state through the intersections of the estimated regions of stability of different controllers to the desired operating point with guaranteed stability. The approach in Reference [8] was also applied to systems with input amplitude and rate saturation [9, 10]. But both approaches in References [7, 8] do not account for constraints explicitly by on-line optimization.

In this paper, we present a computationally efficient scheduled robust constrained MPC that possesses the merits of both NMPC and gain scheduling. The basic idea can be summarized as follows: (1) locally represent a nonlinear system around an equilibrium point as a linear time varying (LTV) system and develop a robust predictive controller with an estimate of its region of stability; (2) expand the region of stability about the desired operating point by piecing together the estimated regions of stability of a set of local predictive controllers. We implement the resulting family of local predictive controllers as a single scheduled MPC which on-line switches between the local controllers and achieves nonlinear transitions with guaranteed stability. This algorithm is computationally efficient, because at each sampling time, the original nonlinear non-convex optimization problem has been reduced to a line search and a convex optimization problem involving LMIs. It also provides a general framework for the scheduled MPC design, which can incorporate any local MPC design scheme once its explicit region of stability is characterized. Among various formulations of NMPC, Cannon et al. [11] proposed an efficient nonlinear predictive controller with its region of stability approximated by an invariant ellipsoidal set, which can be a candidate for the local predictive controllers in the scheduled MPC design proposed in this paper.

Notation. The notation used is fairly standard. The matrix inequality \( A > B \) \((A \geq B)\) means that \( A \) and \( B \) are square symmetric and \( A - B \) is positive (semi-) definite. \( I \) denotes the identity matrix. For a set of scalars/vectors \( \{d^{(i)}\} \) (or matrices \( \{A^{(i)}\} \)), \( d^{(i)} \) (or \( A^{(i)} \)) denotes the \( i \)-th scalar/vector (or matrix). \( x(k) \) or \( x(k|k) \) denotes the state measured at real time \( k \); \( x(k+i|k) \) \((i \geq 1)\) the state at prediction time \( k+i \) predicted at real time \( k \).

2. PREVIOUS WORK

Consider a LTV system

\[
\begin{align*}
    x(k + 1) &= A(k)x(k) + B(k)u(k) \\
    [A(k) & B(k)] \in \Omega
\end{align*}
\]

(1)

where \( u(k) \in \mathbb{R}^m \) is the control input subject to \( |u_r(k)| \leq u_{r,\text{max}}, \ r = 1, 2, \ldots, m \), and \( x(k) \in \mathbb{R}^n \) is the state of the plant subject to \( |x_r(k)| \leq x_{r,\text{max}}, \ r = 1, 2, \ldots, n \). For polytopic uncertainty, \( \Omega \) is the polytope \( \text{Co}\{[A_1, B_1], \ldots, [A_L, B_L]\} \), where \( \text{Co} \) denotes the convex hull, \([A_i, B_i]\) are vertices of the
convex hull. Any \([A B] \) within the convex set \( \Omega \) is a linear combination of the vertices \( A = \sum_{j=1}^{L} x_j A_j, \ B = \sum_{j=1}^{L} x_j B_j \) with \( \sum_{j=1}^{L} x_j = 1, \ 0 \leq x_j \leq 1 \).

The objective is to minimize the worst case infinite horizon quadratic objective function:

\[
\begin{align*}
\min_{u(k+1)=F(k)u(k)+r(k)} \quad & \max_{[A(k+i)B(k+i)]\in\Omega, \ i \geq 0} J_{\\\\infty}(k) \\
\text{subject to} \quad & |u_i(k+i)| \leq u_{r,\text{max}}, \ r = 1, 2, \ldots, m, [x_i(k+i)] \leq x_{r,\text{max}}, \ r = 1, 2, \ldots, n, \ i \geq 0 ,
\end{align*}
\]

where \( J_{\\\\infty}(k) = \sum_{i=0}^{\infty} [x(k+i)\bar{x}(k+i) + u(k+i)^T \mathcal{A} u(k+i)] \) with \( \mathcal{A} > 0, \ \mathcal{B} > 0 \). In (2), we assume that at each sampling time \( k \), a state feedback law \( u(k+i) = F(k)x(k+i) \) is used to minimize the worst case value of \( J_{\\\\infty}(k) \). Following an approach given in [12], it is easy to derive an upper bound on \( J_{\\\\infty}(k) \). At sampling time \( k \), define a quadratic function \( V(x) = x^T P(k)x, \ P(k) > 0 \). For any \([A(k+i) B(k+i)] \in \Omega, \ i \geq 0 \), suppose \( V(x) \) satisfies the following robust stability constraint:

\[
V(x(k+i+1) - V(x(k+i)) \leq -(x(k+i+1)^T \mathcal{A} x(k+i) + u(k+i+1)^T \mathcal{B} u(k+i+1))
\]

(3) and (4) give an upper bound on \( J_{\\\\infty}(k) \). The condition \( V(x(k+i)) \leq \gamma \) in (4) can be expressed equivalently as the LMIs

\[
\begin{bmatrix}
1 & x(k+i)^T \\
\end{bmatrix} \geq 0, \quad Q > 0
\]

where \( Q = \gamma P(k)^{-1} \). The robust stability constraint (3) is satisfied if for each vertex of \( \Omega \) [12]

\[
\begin{bmatrix}
Q & QA^T_J + Y^TB^T_J & \mathcal{B}^{1/2} \\
A_J Q + B_J Y & \gamma I & 0 \\
\mathcal{B}^{1/2} Q & 0 & \gamma I \\
\end{bmatrix} \geq 0, \quad \gamma I = 0
\]

(6)

where, \( Q = \gamma P(k)^{-1} \) and \( F(k) \) is parameterized by \( YQ^{-1} \). The input constraints are satisfied if there exists a symmetric matrix \( X \) such that [12]

\[
\begin{bmatrix}
X & Y \\
Y^T & Q \\
\end{bmatrix} \geq 0 \quad \text{with} \quad X_{rr} \leq u_{r,\text{max}}, \ r = 1, 2, \ldots, m
\]

(7)

Similarly, the state constraints are satisfied if there exists a symmetric matrix \( Z \) such that

\[
Z - Q \geq 0 \quad \text{with} \quad Z_{rr} \leq x_{r,\text{max}}, \ r = 1, 2, \ldots, n
\]

(8)

**Theorem 1 (Robust constrained MPC Kothare et al. [12])**

For the system (1), at sampling time \( k \), the state feedback matrix \( F(k) \) in the control law \( u(k+i) = F(k)x(k+i) \) \( i \geq 0 \), which minimizes the upper bound \( \gamma \) on the worst case MPC objective function \( J_{\\\\infty}(k) \), is given by \( F(k) = YQ^{-1} \) where \( Q > 0 \) and \( Y \) are obtained from the solution (if it exists) of the linear objective minimization problem \( \min_{\gamma, Q, X, Y, Z} \gamma \) subject to (5)–(8).
Suppose this MPC algorithm is initially feasible, then it robustly asymptotically stabilizes the closed-loop system.

3. SCHEDULED ROBUST CONSTRAINED MPC

We consider discrete-time nonlinear dynamical systems described by

$$x(k + 1) = f(x(k), u(k))$$  \hspace{1cm} (9)

where $x(k) \in X \subseteq \mathbb{R}^n$ and $u(k) \in U \subseteq \mathbb{R}^m$ are the system state and control input, respectively, $X$ and $U$ are compact sets. Assume

$$f(x) = \begin{bmatrix} f_1(x, u) \\ \vdots \\ f_n(x, u) \end{bmatrix}$$

are continuous differentiable.

**Definition 1**

Given a set $U$, a point $x_0 \in X$ is an equilibrium point of the system (9) if a control $u_0 \in \text{int}(U)$ exists such that $x_0 = f(x_0, u_0)$. We call a connected set of equilibrium points an equilibrium surface.

Suppose $(x^{eq}, u^{eq})$ is a point on an equilibrium surface. Define a shifted state $\tilde{x} = x - x^{eq}$ and a shifted input $\tilde{u} = u - u^{eq}$, the nonlinear system with respect to $(x^{eq}, u^{eq})$ can be expressed as

$$\tilde{x}(k + 1) = f(x(k), u(k)) - f(x^{eq}, u^{eq}) = g(\tilde{x}(k), \tilde{u}(k))$$  \hspace{1cm} (10)

3.1. Robust constrained MPC with an estimated region of stability

Within a neighbourhood around $(x^{eq}, u^{eq})$, i.e. $\Pi_x = \{ x \in \mathbb{R}^n | |x_r - x^{eq}_r| \leq \delta x_r, r = 1, \ldots, n \} \subseteq X$, and $\Pi_u = \{ u \in \mathbb{R}^m | |u_r - u^{eq}_r| \leq \delta u_r, r = 1, \ldots, m \} \subseteq U$, we locally represent the nonlinear system (10) within the neighbourhood as a LTV system

$$\tilde{x}(k + 1) = A(k)\tilde{x}(k) + B(k) \tilde{u}(k)$$

$$\begin{bmatrix} A(k) & B(k) \end{bmatrix} \in \Omega$$  \hspace{1cm} (11)

subject to $|\tilde{u}_r(k + i|k)| \leq \delta u_r, r = 1, \ldots, m, |\tilde{x}_r(k + i|k)| \leq \delta x_r, r = 1, \ldots, n, i \geq 0$. $\Omega$ can be described as polytopic uncertainty such that for all $\tilde{x} \in \Pi_x$ and $\tilde{u} \in \Pi_u$, the Jacobian matrix

$$\frac{\partial g}{\partial \tilde{x}} \frac{\partial g}{\partial \tilde{u}} \in CO \Omega$$  \hspace{1cm} (12)

with $\frac{\partial g}{\partial \tilde{x}} = \begin{bmatrix} \frac{\partial g}{\partial \tilde{x}_1} & \cdots & \frac{\partial g}{\partial \tilde{x}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial \tilde{x}_1} & \cdots & \frac{\partial g}{\partial \tilde{x}_n} \end{bmatrix}$ and $\frac{\partial g}{\partial \tilde{u}} = \begin{bmatrix} \frac{\partial g}{\partial \tilde{u}_1} & \cdots & \frac{\partial g}{\partial \tilde{u}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial \tilde{u}_1} & \cdots & \frac{\partial g}{\partial \tilde{u}_n} \end{bmatrix}$

When we apply the robust constrained MPC in Theorem 1 to the LTV system (11), an ellipsoidal region of feasibility can be characterized by the following lemma.

**Lemma 1**
For the LTV system (11), an ellipsoidal region of feasibility of the robust constrained MPC in Theorem 1 is given by
\[
S = \{ \mathbf{x} \in \mathbb{R}^n | \mathbf{x}^T R^{-1} \mathbf{x} \leq 1 \},
\]
where \( R \) is the optimal solution \( Q \) of the maximization problem \( \max_{Q, X, Y, Z} \log \det(Q) \) subject to (6)–(8).

**Proof**
When the robust constrained MPC in Theorem 1 is applied to a state \( \mathbf{x} \) of the system (11), the only LMI that depends on the system state is (5) which is automatically satisfied for all states within the ellipsoid \( \mathcal{E} = \{ \mathbf{x} \in \mathbb{R}^n | \mathbf{x}^T Q^{-1} \mathbf{x} \leq 1 \} \). So \( \mathcal{E} \) defines a feasible region of the robust constrained MPC. The maximization in Lemma 1 maximizes the volume of the ellipsoid \( \mathcal{E} \), which is proportional to \( \det(Q)^{1/2} \), and therefore maximizes the ellipsoidal feasible region of the robust constrained MPC.

**Remark 1**
The larger the neighbourhood around the equilibrium point, the larger the uncertainty of the LTV system. The maximum size of the neighbourhood in which the optimization in Lemma 1 is feasible is a measure of process nonlinearity around the equilibrium point.

**Remark 2**
For a large neighbourhood, the ellipsoidal feasible region obtained by Lemma 1 may not be large (see Example 2 Figure 4). So we try to select a neighbourhood such that the ellipsoidal feasible region obtained by Lemma 1 is an inner approximation of \( \Pi_x \) that touches the boundary of \( \Pi_x \).

Replacing the state constraints by \( \mathbf{x}(k + i|k) \in \mathcal{E} \subset \Pi_x, \ i \geq 0 \), or, equivalently
\[
R - Q > 0 \quad (13)
\]
which confines the current state and all future predicted states to be inside the feasible region \( \mathcal{E} \), we develop a robust constrained MPC with an estimate of its region of stability.

**Algorithm 1**
For system (11), at sampling time \( k \), apply
\[
u(k) = F(k) x(k)
\]
where \( F(k) = Y Q^{-1} \) with \( Q > 0 \) and \( Y \) obtained from the minimization problem \( \min_{Q, X, Y, Z} \gamma \) subject to (5)–(7) and (13), where \( R \) is obtained from Lemma 1.

In order to show that the robust constrained MPC in Algorithm 1 has an estimated region of stability \( \mathcal{E} = \{ \mathbf{x} \in \mathbb{R}^n | \mathbf{x}^T R^{-1} \mathbf{x} \leq 1 \} \) for both the LTV system (11) and the nonlinear system (10), we first introduce a concept of the asymptotically stable invariant ellipsoid.

**Definition 2** (Wan and Kothare [13])
Given a discrete dynamical system \( x(k + 1) = f(x(k)) \), a subset \( \mathcal{E} = \{ x \in \mathbb{R}^n | x^T Q^{-1} x \leq 1, \ Q > 0 \} \) of the state space \( \mathbb{R}^n \) is said to be an asymptotically stable invariant ellipsoid, if it has the property that, whenever \( x(k_1) \in \mathcal{E} \), then \( x(k) \in \mathcal{E} \) for all times \( k \geq k_1 \) and \( x(k) \to 0 \) as \( k \to \infty \).
Lemma 2
Consider a closed-loop system composed of the LTV system (11) and a robust constrained MPC in Algorithm 1. Then, the subset $E = \{ \bar{x} \in \mathbb{R}^n | \bar{x}^T R^{-1} \bar{x} \leq 1 \}$ of the state space $\mathbb{R}^n$ is an asymptotically stable invariant ellipsoid.

Proof
Consider the LTV system (11), for any $\bar{x}(k) \in \mathcal{S}$, the robust constrained MPC in Algorithm 1 is guaranteed to be feasible, and the satisfaction of (6) and (13) implies that $\bar{x}(k+1) \in \mathcal{S}$ in real time. So $\mathcal{S}$ is an invariant set. Furthermore, initial feasibility implies asymptotic stability (using Theorem 1). Thus, $\bar{x}(k) \to 0$ as $k \to \infty$. This establishes that $\mathcal{S}$ is an asymptotically stable invariant set.

Theorem 2
For the nonlinear system (10), the robust constrained MPC in Algorithm 1 can asymptotically stabilize the closed-loop system with an estimated region of stability $\mathcal{S} = \{ \bar{x} \in \mathbb{R}^n | \bar{x}^T R^{-1} \bar{x} \leq 1 \}$.

Proof
Since the LTV system (11) is a representation of a class of nonlinear system satisfying (12) (including the given nonlinear system (10) within the neighbourhood $(\Pi_1, \Pi_0)$), it is straightforward from Lemma 2 that $\mathcal{S}$ is also an asymptotically stable invariant set for the closed-loop system composed of the given nonlinear system (10) and the robust constrained MPC in Algorithm 1.

The region of stability of a discrete system is defined as a subset $D \subset \mathbb{R}^n$ such that for any $x(0) \in D$, $x(k) \to 0$ as $k \to \infty$. So $\mathcal{S}$ is an estimate of the region of stability for the nonlinear closed-loop system.

The invariance property of $\mathcal{S}$ is important, because it guarantees that all trajectories of the nonlinear system (10) starting within $\mathcal{S}$ never go outside of the ellipsoid, thus making the LTV system (11) a valid representation of the original nonlinear system (10) within $\mathcal{S}$.

Remark 3
Within the neighbourhood, the nonlinear system can also be described as norm-bound uncertainty [11], and robust MPC for the LTV system with norm-bound uncertainty [12] can be applied. Since the LTV model representation with norm-bound uncertainty has much smaller size than that with polytopic uncertainty, the former is more computationally efficient for higher order nonlinear systems.

Remark 4
The advantages of using the robust constrained MPC in Algorithm 1 for control of constrained nonlinear systems are as follows:

1. The original nonlinear non-convex optimization problem has been reduced to a convex optimization problem involving LMIs. Further computation reduction can be achieved by using the off-line formulation of robust constrained MPC [13].

2. The robust MPC in Algorithm 1 has the region of stability $\mathcal{S}$ not only for the given nonlinear system but also for a class of nonlinear systems that satisfy (12), therefore it is more robust with respect to model uncertainty.
3. Exponential stability and exponential decay of the performance cost can be achieved by imposing an additional constraint

\[
\begin{bmatrix}
p^2Q & QA_j^T + Y^TB_j^T \\
A_jQ + B_jY & Q
\end{bmatrix} \succeq 0, \quad 0 < p < 1, \quad j = 1, \ldots, L
\]

(14)
in both the maximization in Lemma 1 and the minimization in Algorithm 1. (see Reference [12, remark 7]).

4. Since the ellipsoidal feasible region \(\mathcal{S}\) obtained by Lemma 1 with (or without) the additional constraint (14) also provides a feasible region for various NMPC algorithms, e.g. the contractive NMPC (see Reference [6, Remark 4]), the nonlinear non-convex optimizations of these NMPC algorithms can be solved in \(\mathcal{S}\) with guaranteed feasibility.

Remark 5
The performance conservatism of the worst case performance optimization in Algorithm 1 is discussed as follows.

1. Consider the nonlinear system (10) and a state close to the equilibrium. Comparing to a controller that on-line solves the original nonlinear non-convex optimization, the performance conservatism of Algorithm 1 in the worst case can be quantified as the difference between the worst case performance and the nominal performance solved by Algorithm 1 based on the LTV model (11) and the linearized model at the equilibrium point, respectively.

2. In case that the nonlinear system is of the form \(x(k + 1) = f(x(k)) + Bu(k)\) where the LTV model does not depend on \(\Pi_u\), the symmetric input constraints specified in \(\Pi_u\) are likely to be conservative around the equilibrium points for which the input constraint \(u(k) \in U\) is asymmetric with respect to \(u^{eq}\).

3.2. Scheduled robust constrained MPC
Given an equilibrium surface, we can design a robust predictive controller with an estimate of its region of stability around any equilibrium point in the same manner as for \((x^{eq}, u^{eq})\) in the previous subsection, provided the linearized dynamics about this point can be stabilized. We first design Controller \#0 with its estimated region of stability \(\mathcal{S}(0)\) for the desired equilibrium point \((x(0), u(0))\). Let \(i := 0\). Since by definition, the equilibrium surface is a connected set, we can always find another equilibrium point \((x(i+1), u(i+1))\) satisfying \(x(i+1) \in \mathcal{S}(0)\), and design Controller \#(i + 1) with its estimated region of stability \(\mathcal{S}(i+1)\) for \((x(i+1), u(i+1))\). In this manner, we keep constructing controllers until the union of their estimated regions of stability covers the desired operating region.

On-line, we implement the resulting family of robust predictive controllers as a single controller whose parameters are changed by monitoring the scheduling variable \(i\). The scheduling variable \(i\) can provide information about which region of stability the current state is in and which controller parameters to use. We call such a controller scheme a scheduled robust constrained MPC.

Algorithm 2 (Scheduled robust constrained MPC)
For the system (9), given an equilibrium surface and a desired equilibrium point \((x(0), u(0))\). Let \(i := 0\).
1. Specify a neighbourhood \((\Pi^i_0, \Pi^i_u)\) around \((\bar{x}^i, u^i)\) satisfying \(\Pi^i_x \subseteq X\) and \(\Pi^i_u \subseteq U\), define \(\bar{x} = x - \bar{x}^i\) and \(\bar{u} = u - u^i\), and locally represent the nonlinear system \(10\) within the neighbourhood as a LTV system \(11\).

2. Design Controller \#i (Algorithm 1) with an estimate of its region of stability

\[
\mathcal{S}^{(i)} = \{x \in \mathbb{R}^n | (x - \bar{x}^i)^T (R^{(i)})^{-1} (x - \bar{x}^i) \leq 1\}
\]

Store \((\bar{x}^i, u^i, (R^{(i)})^{-1})\) in a lookup table;

3. Select an equilibrium point \((\bar{x}^{i+1}, u^{i+1})\) satisfying \(x^{i+1} \in \mathcal{S}^{(i)}\). Let \(i := i + 1\) and go to step 1, until the region \(\bigcup_{i=0}^{M} \mathcal{S}^{(i)}\) with \(M = \max i\) covers a desired portion of the equilibrium surface.

On-line, given an initial state \(x(0) \in \bigcup_{i=0}^{M} \mathcal{S}^{(i)}\). Let the state be \(x(k)\) at time \(k\). Perform a line search over \((\bar{x}^i, u^i, (R^{(i)})^{-1})\) in the lookup table to find the smallest index \(i\) such that \(x(k) \in \mathcal{S}^{(i)}\). Implement \(u(k) = F^{(i)}(k)(x(k) - \bar{x}^i) + u^i\), where \(F^{(i)}(k)\) is obtained from Controller \#i.

Remark 6

Since by assumption the equilibrium point is locally stabilizable, it is guaranteed that there exists a neighbourhood around the equilibrium point such that step 2 is feasible.

Theorem 3

For the nonlinear system \((9)\), the scheduled robust constrained MPC in Algorithm 2 has an estimated region of stability \(\bigcup_{i=0}^{M} \mathcal{S}^{(i)}\) around the desired equilibrium \((\bar{x}^0, u^0)\).

Proof

The condition \(x^{i+1} \in \mathcal{S}^{(i)}, i = 0, \ldots, M - 1\), ensures that as trajectories starting from \(\mathcal{S}^{(i+1)}\) converge to \(x^{i+1}\), a switching point always exists over the intersection \(\mathcal{S}^{(i+1)} \cap \mathcal{S}^{(i)}\) where Controller \#(i + 1) can be switched to Controller \#i.

On-line search for the controller with the smallest index \(i\) such that \(x \in \mathcal{S}^{(i)}\) is equivalent to the switching rule that once \(x \in \mathcal{S}^{(i+1)} \cap \mathcal{S}^{(i)}\), switch from Controller \#(i + 1) to Controller \#i. Given a dynamical system \((1)\) and an initial state \(x(0) \in \bigcup_{i=0}^{M} \mathcal{S}^{(i)}\), the closed-loop system becomes

\[
x(k + 1) = \begin{cases} 
    f(x(k), F^{(i)}(k)(x(k) - \bar{x}^i) + u^i) & \text{if } x(k) \in \mathcal{S}^{(i)} \setminus \mathcal{S}^{(i-1)}, \; i \neq 0 \\
    f(x(k), F^{(0)}(k)(x(k) - \bar{x}^0) + u^0) & \text{if } x(k) \in \mathcal{S}^{(0)} 
\end{cases}
\]

When \(x(k) \in \mathcal{S}^{(i)} \setminus \mathcal{S}^{(i-1)}, \; i \neq 0\), the control law \(u(k) = F^{(i)}(k)(x(k) - \bar{x}^i) + u^i\) is guaranteed to keep the state within \(\mathcal{S}^{(i)}\) and converge to \(x^i \in \mathcal{S}^{(i-1)}\). As soon as \(x \in \mathcal{S}^{(i)} \cap \mathcal{S}^{(i-1)}\), switch from Controller \#i to Controller \#(i - 1) with its operating region \(\mathcal{S}^{(i-1)} \setminus \mathcal{S}^{(i-2)}\), and so on. Lastly, \(x \in \mathcal{S}^{(0)}\), Controller \#0 is guaranteed to keep the state within \(\mathcal{S}^{(0)}\) and converge to \(x^0\).

Remark 7

The region of stability of the scheduled MPC can be expanded in different directions over the given equilibrium surface, creating multiple branches of the region of stability. By adding another dimension of the index of the controller, such as Controller \#ij with \(i\) the branch index and \(j\) the controller index, Algorithm 2 can be used to expand the region of stability along multiple directions over the given equilibrium surface.
Remark 8
In general, the scheduled MPC in Algorithm 2 requires a specified path on the equilibrium surface so as to extend the region of stability. For the same path on the equilibrium surface, the larger the number of controllers designed, the more overlap between the estimated regions of stability of two adjacent controllers, and the better the transition performance, because controller switches can happen without moving the state trajectory close to the intermediate equilibrium points. Yet a larger number of controllers leads to a larger storage space for the lookup table and a longer time to do the search. So there is a trade-off between achievement of good transition performance and computational efficiency.

Corollary 1
Consider $M + 1$ local predictive controllers designed by Algorithm 2. Suppose not only $x^{(i)} \in \mathcal{S}^{(i-1)}$, $i = 1, \ldots, M$, but also $x^{(i)} \in \mathcal{S}^{(i+1)}$, $i = 0, \ldots, M - 1$. Given an initial state $x(0) \in \bigcup_{i=0}^{M} \mathcal{S}^{(i)}$ and $j \in \{0, \ldots, M\}$. Let the state be $x(k)$ at time $k$. If Controller $#i^\ast = \arg\min[j - j]$ subject to $x(k) \in \mathcal{S}^{(i)}$, then as $k \to \infty$, $x(k) \to x^{(0)}$.

Remark 9
In Algorithm 2, if $x^{(i+1)}(i = 0, \ldots, M - 1)$ is chosen sufficiently close to $x^{(i)}$, the estimated region of stability $\mathcal{S}^{(i+1)}$ of Controller $#(i + 1)$ will encompass $x^{(i)}$.

Remark 10
In the presence of certain norm bound asymptotically vanishing disturbance, Algorithm 2 composed of the exponentially stable predictive controllers (see Remark 4 Item 3) can still asymptotically stabilize the closed-loop system to the desired equilibrium $(x^{(0)}, u^{(0)})$.

Remark 11
With minor modifications, Algorithm 2 can handle a nonvanishing norm-bound disturbance, which lumps together various uncertain terms due to model simplification, parameter uncertainty, persistent noise and so on. Because the disturbance is merely bounded, the most that can be achieved is to steer the state to a set instead of the equilibrium point. So in the presence of the nonvanishing norm-bound disturbance, the asymptotically stable invariant ellipsoid $\mathcal{S}$ is redefined such that whenever $x(k_1) \in \mathcal{S}$, then $x(k) \in \mathcal{S}$ for all times $k \geq k_1$ and $x(k) \to \mathcal{S}_0 \subset \mathcal{S}$ as $k \to \infty$. And Step 3 in Algorithm 2 requires $\mathcal{S}^{(i+1)} \subset \mathcal{S}^{(i)}$ instead of $x^{(i+1)} \in \mathcal{S}^{(i)}$.

For the state outside of the estimated region of stability of the scheduled MPC, we can use free control moves to steer it into $\bigcup_{i=0}^{M} \mathcal{S}^{(i)}$, where the scheduled MPC can be applied.

Corollary 2
Consider the scheduled MPC in Algorithm 2 with its estimated region of stability $\bigcup_{i=0}^{M} \mathcal{S}^{(i)}$ covering the desired portion of a given equilibrium surface. On-line, given $x(0) \notin \bigcup_{i=0}^{M} \mathcal{S}^{(i)}$ and $N > 0$. Let $k := 0$.

1. Solve the following finite horizon optimization problem:

$$\min_{u(k), \ldots, u(k+N-1)} \sum_{i=0}^{N-1} [x(k + ik)^T 2x(k + ik) + u(k + ik)^T R u(k + ik)]$$

(15)
subject to (9) and the terminal constraint

$$x(k + N|k) \in \bigcup_{i=0}^{M} \mathcal{G}(i)$$

(16)

If feasible, implement $u(k|k)$; Otherwise, let $N := N + c$, $c > 0$, repeat Step 1.

2. At time $k + 1$. If $x(k + 1) \in \bigcup_{i=0}^{M} \mathcal{G}(i)$, apply the scheduled MPC in Algorithm 2; Otherwise, if $N > 1$, let $N := N - 1$; if $N = 1$, let $N := N$. Let $k := k + 1$, repeat Step 1.

Suppose there are no persistent disturbances and Step 1 is feasible at each sampling time for some finite $N$, then the state is guaranteed to converge to $\bigcup_{i=0}^{M} \mathcal{G}(i)$ in finite time.

**Proof**

If there are no disturbances, the prediction of the optimization at time $k$ is exact, which means that if the finite $N$ control moves obtained at time $k$ are implemented in real time from $k$ to $k + N - 1$, $x(k + N) \in \bigcup_{i=0}^{M} \mathcal{G}(i)$ will be satisfied. So after $u(k|k)$ is implemented, the remaining $N - 1$ control moves provide a feasible solution for the optimization with a control horizon $N - 1$ at time $k + 1$. Using the above argument repeatedly, we can show that the optimizations solved from $k$ to $k + N - 1$ with the control horizons strictly decreasing from $N$ to 1 are all feasible. Finally, after the last optimization with $N = 1$ is solved and the current control move implemented, the state is brought into $\bigcup_{i=0}^{M} \mathcal{G}(i)$. In case there are asymptotically vanishing disturbances, provided that Step 1 is feasible at each sampling time, after sufficiently large time $k > 0$, the disturbances approach zero, and the above arguments can be used to show the convergence of the state to $\bigcup_{i=0}^{M} \mathcal{G}(i)$.

**Remark 12**

The proposed algorithm in Corollary 2 adopts the variable horizon dual mode NMPC paradigm, whose many advantages over a fixed horizon NMPC were summarized in Reference [2]. In fact, the major advantage of the variable horizon NMPC in Corollary 2 is that a global optimal solution is not required and any feasible solution of the optimization problem (15) is an acceptable solution for Corollary 2 to guarantee the convergence of the state to $\bigcup_{i=0}^{M} \mathcal{G}(i)$. Robustness can be enhanced by requiring the terminal state to enter $\bigcup_{i=0}^{M} \mathcal{G}(i)$ with $\mathcal{G}(i) = \{x \in \mathbb{R}^n | (x - x^{(i)})^T (R^{(i)})^{-1} (x - x^{(i)}) \leq \theta^{(i)} < 1\}$ in the same way as in Reference [2]. Furthermore, we can easily convert the variable horizon MPC in Corollary 2 into a fixed horizon MPC with the terminal constraint set $\mathcal{G}(i)$ and the terminal cost $x(k + N|k)^T \gamma^{(i)} (R^{(i)})^{-1} x(k + N|k)$, where $\gamma^{(i)}$ and $R^{(i)}$ are obtained from Lemma 1 for Controller #i.

4. EXAMPLES

In this section, we present two examples that illustrate the implementation of the proposed scheduled MPC. For both these examples, the LMI Control Toolbox and the Optimization Toolbox in the MATLAB environment were used to solve the linear minimization problem in the proposed scheduled MPC and the nonlinear optimization in Corollary 2, respectively.
4.1. Example 1

Consider a two-tank system [14]

\[
\begin{align*}
\rho S_1 \dot{h}_1 &= -\rho A_1 \sqrt{2gh_1} + u \\
\rho S_2 \dot{h}_2 &= \rho A_1 \sqrt{2gh_1} - \rho A_2 \sqrt{2gh_2}
\end{align*}
\]

(17)

where \( \rho = 0.001 \text{ kg/cm}^2, \ g = 980 \text{ cm/s}^2, \ S_1 = 2500 \text{ cm}^2, \ A_1 = 9 \text{ cm}^2, \ S_2 = 1600 \text{ cm}^2, \ A_2 = 4 \text{ cm}^2, \ 1 \text{ cm} \leq h_1 \leq 50 \text{ cm}, \ 10 \text{ cm} \leq h_2 \leq 120 \text{ cm}, \ \text{and} \ 0 \leq u \leq 2.5 \text{ kg/s}. \) The equilibrium surface is determined using the steady-state version of (17) and is shown as the solid line in Figure 1. Consider an equilibrium point \((h_1^{eq}, h_2^{eq})^T, u^{eq})\), locally represent the nonlinear system (17) within a neighbourhood \((\Pi_h, \Pi_u)\) as a LTV system. \( \Omega \) can be described as polytopic uncertainty with four vertices \(\{J(h_1^{eq} + \delta h_1, h_2^{eq} + \delta h_2), J(h_1^{eq} + \delta h_1, h_2^{eq} - \delta h_2), J(h_1^{eq} - \delta h_1, h_2^{eq} + \delta h_2), J(h_1^{eq} - \delta h_1, h_2^{eq} - \delta h_2)\}\), where \(J(h_1, h_2)\) is the Jacobian matrix at \((h_1, h_2)^T\). The sampling time is 0.5 sec, and the design parameters \(\mathcal{S} = \text{diag}(0, 1)\) and \(\mathcal{R} = 0.01\).

Figure 1 shows the estimated regions of stability of the scheduled MPC #1 composed of Controller #0, 1, 2, 3 and the scheduled MPC #2 composed of Controller #0, 1, 2', 3, with \(\mathcal{S}^{(2)}\) shown in dotted line. Here, we apply the embedding approach [14] with \(N = 0\), and design Controller #2' with an estimate of its region of stability within the state constraints.

Consider the regulation from an initial state \(x(0) = [5 \ 15]^T\) to the equilibrium \((x^{(0)}, u^{(0)}) = ([19.753 \ 100]^T, 1.7710)\). Figure 1 shows the phase plots of the regulations by the scheduled MPC #1 and #2, where a switching from Controller \#(i + 1) to Controller \#i happens at \(\mathcal{S}^{(i+1)} \cap \mathcal{S}^{(i)}\) with \(\partial \mathcal{S}^{(i)}\) denoting the boundary of \(\mathcal{S}^{(i)}\), \(i = 0, 1, 2\) (or 2'). From Figure 1, we can also see that the embedding controller #2' itself cannot achieve transition from \([5, 15]^T\) to \([19.7, 100]^T\), it can only regulate the states within its estimated region of stability (dotted line) to the equilibrium \([12.8, 64.9]^T\). Figure 2 shows the time responses of the system states which are

Figure 1. Phase plots of the regulations from \(x(0) = (5, 15)^T\) to the equilibrium ((19.753, 100)^T, 1.7710). Scheduled MPC #1 (solid line), scheduled MPC #2 (dashed line).
transferred over the large operating region with the satisfaction of both input and state constraints and guaranteed stability. The scheduled MPC \#2 performs better than the scheduled MPC \#1, because Controller \#2 offers larger estimated region of stability than Controller \#2 provides larger control moves. On a Gateway PC with Pentium III processor (1000 MHz, Cache RAM 256 KB and total memory 256 MB) and using Matlab code, the numerical complexity of the proposed scheduled MPC resulted in 5811 flops per step similar to what was reported for the embedding approach with \( N = 0 \) in Reference [14].

4.2. Example 2

Consider the highly nonlinear model of a continuous stirred tank reactor (CSTR) [4]. Assuming constant liquid volume, the CSTR for an exothermic, irreversible reaction, \( A \rightarrow B \), is described by the following dynamic model based on a component balance for reactant \( A \) and an energy balance:

\[
\begin{align*}
\dot{C}_A &= \frac{q}{V} (C_{Af} - C_A) - k_0 \exp\left(-\frac{E}{RT}\right) C_A \\
\dot{T} &= \frac{q}{V} (T_f - T) + \frac{(\Delta H)_{r\text{xn}}}{\rho C_p} k_0 \exp\left(-\frac{E}{RT}\right) C_A + \frac{UA}{V \rho C_p} (T_c - T)
\end{align*}
\]  

(18)

where \( C_A \) is the concentration of \( A \) in the reactor, \( T \) is the reactor temperature, and \( T_c \) is the temperature of the coolant stream. The parameters are \( q = 100 \text{ l/min} \), \( V = 100 \text{ l} \), \( C_{Af} = 1 \text{ mol/l} \), \( T_f = 400 \text{ K} \), \( \rho = 10^3 \text{ g/l} \), \( C_p = 1 \text{ J/(g K)} \), \( k_0 = 4.71 \times 10^8 \text{ min}^{-1} \), \( E/R = 8000 \text{ K} \), \( \Delta H_{r\text{xn}} = -2 \times 10^5 \text{ J/mol} \), \( UA = 10^5 \text{ J/(min K)} \), \( 0 \leq C_A \leq 1 \text{ mol/l} \), \( 250 \text{ K} \leq T \leq 500 \text{ K} \), and \( 250 \text{ K} \leq T_c \leq 500 \text{ K} \). The equilibrium surface is determined using the steady-state version of (18) and is

Figure 2. Time responses of the regulations from \( x(0) = (5, 15)^T \) to the equilibrium \((19.753, 100)^T, 1.7710\). Scheduled MPC \#1 (solid line), scheduled MPC \#2 (dashed line).
shown as the solid curve in Figure 3. Consider an equilibrium point

\[
\begin{bmatrix}
C_{\text{eq}} \\
T_{\text{eq}}
\end{bmatrix}
\]

locally represent the nonlinear system (18) within a neighbourhood \((\Pi_x, \Pi_u)\) as a LTV system. \(\Omega\) can be described as polytopic uncertainty with four vertices 

\[
\{J(C_{\text{eq}} + \delta C_A, T_{\text{eq}} + \delta T), J(C_{\text{eq}} - \delta C_A, T_{\text{eq}} + \delta T), J(C_{\text{eq}} - \delta C_A, T_{\text{eq}} - \delta T), J(C_{\text{eq}} + \delta C_A, T_{\text{eq}} - \delta T)\},
\]

where \(J(C_A, T)\) is the Jacobian matrix at \((C_A, T)\). The sampling time is 0.03 min, and the design parameters \(\mathcal{R} = \text{diag}(\frac{1}{0.35}, \frac{1}{400})\) and \(\mathcal{R} = \frac{1}{300}\).

Under the nominal operating condition \(T_c = 302\) K, the reactor exhibits three equilibria, i.e. \((C_A \text{eq}, T_{\text{eq}}) = (0.899, 361.141)\text{T}, (0.52, 398.972)\text{T}\) and \((0.189, 432.083)\text{T}\). The first and third equilibria are stable, while the second equilibrium is unstable but desirable. The objective is to transfer the system from the first or the third equilibrium to the second equilibrium. Figure 4 shows that for Controller #0 when we select a larger neighbourhood (dotted line), we get a smaller ellipsoidal feasible region (dotted line) from Lemma 1. So we choose the neighbourhood (solid line) such that the ellipsoidal feasible region (solid line) touches the boundary of the neighbourhood. Figure 3 shows the phase plots of the transitions from \((0.899, 361.141)\text{T}\) to \((0.52, 398.972)\text{T}\) by using six local predictive controllers, and from \((0.189, 432.083)\text{T}\) to \((0.52, 398.972)\text{T}\) by using ten local predictive controllers. Figure 5 shows the time responses for the transitions.

Furthermore, consider the six local controllers in branch #1, according to Corollary 1, the scheduled MPC composed of the six local predictive controllers can regulate any initial state \((C_A(0), T(0))\text{T}\in\bigcup_{j=0}^5 \mathcal{G}^{(j)}\) to any equilibrium point \(((C_A^{(j)}, T^{(j)})\text{T}, T_c^{(j)}), j = 0, \ldots, 5\). As shown in Figure 6, the system with an initial state \((0.35, 400)\text{T}\in\mathcal{G}^{(0)}\) has been moved to equilibrium point

\[\text{Figure 3. Phase plots of the transitions from } (0.899, 361.141)\text{T} \text{ to } (0.52, 398.972)\text{T}, \text{ and from } (0.189, 432.083)\text{T} \text{ to } (0.52, 398.972)\text{T}.\]
#2 and #5, respectively. And also in Figure 6, the initial state $(0.53, 360)^T$ is brought into $\mathcal{S}^{(5)}$ by using Corollary 2 with five free control moves, and then regulated to the desired equilibrium point #0 by using the scheduled MPC. At the first 5 sampling times, the control horizons of Corollary 2 strictly decrease from 5 to 1, and the flops to get the five free control moves are $1.893 \times 10^3$, $1.144 \times 10^2$, $0.789 \times 10^3$, $0.276 \times 10^3$, $0.133 \times 10^3$, respectively. After that, 5811 flops per step are required for the proposed scheduled MPC to compute a feedback gain. Note that the nonlinear MPC algorithms in References [3, 4] reported the flops per step in the order of $10^8$. 

Figure 4. The neighbourhoods and the estimated regions of stability.

Figure 5. Time responses for the transitions from $(0.899, 361.141)^T$ to $(0.52, 398.972)^T$, and from $(0.189, 432.083)^T$ to $(0.52, 398.972)^T$. 

5. CONCLUSIONS

In this paper, we have proposed a computationally efficient scheduled MPC formulation for constrained nonlinear systems with large operating regions. Since we were able to characterize explicitly an estimate of the region of stability of the designed local predictive controller, we could expand it by designing multiple predictive controllers. The resulting family of local controllers was then shown to be implementable as a single scheduled MPC which on-line switches between the local controllers and achieves nonlinear transitions with guaranteed stability. This algorithm is computationally efficient and provides a general framework for the scheduled MPC design. We have shown that this scheduled MPC is easily implementable by applying it to a two-tank system and a highly nonlinear CSTR process.

ACKNOWLEDGEMENTS

Partial financial support for this research from the American Chemical Society’s Petroleum Research Fund (ACS-PRF) and from the P. C. Rossin Assistant Professorship at Lehigh University is gratefully acknowledged.

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