Robust Constrained Model Predictive Control for Nonlinear Systems: A Comparative Study

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Abstract

We compare two approaches to controlling nonlinear systems using model predictive control (MPC) techniques. In the first approach, we use an inner feedback loop to linearize the nonlinear plant and use the resulting linear model to synthesize a controller strategy based on standard MPC techniques. The nonlinear constraints resulting from the feedback linearization are handled using an iterative method. In the second approach, we approximate the nonlinear system by a linear time-varying (LTV) system and design a stabilizing receding horizon state-feedback control law using optimization techniques based on linear matrix inequalities (LMIs).

1 Introduction

Realistic models of commonly encountered engineering systems are often nonlinear and must satisfy control constraints. One approach to controlling constrained nonlinear systems is to use an inner feedback loop to linearize the nonlinear plant [6, 5]. The resulting linear model can then be used to synthesize a control strategy based on model predictive control (MPC) or receding horizon control (RHC) techniques. The nonlinear, state-dependent constraints resulting from the feedback linearization (FL) are handled using some iterative or approximation methods. A second approach is to approximate the nonlinear system by an uncertain linear time-varying (LTV) system. For certain classes of uncertain LTV systems, a robustly stabilizing, constrained receding horizon state-feedback controller synthesis has been presented in [2, 3] using optimization techniques based on linear matrix inequalities (LMIs).

The goal of this paper is to analyze the robustness of the FL-MPC approach to uncertainty in the nonlinear plant model by comparing its performance with that of robust LMI-based MPC techniques. The paper is organized as follows: In §2, we outline the key features of the two control approaches. In §3, we present an example to illustrate the two synthesis techniques. In §4, we present concluding remarks.

2 Controller synthesis

We consider the following discrete-time nonlinear system

\[ x(k+1) = f(x(k), u(k)) \]

\[ y(k) = h(x(k)) \]

(1)

Here, \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \) are the state and output of the system and \( u \in \mathbb{R}^p \) is the control input. A typical MPC algorithm solves the following optimization problem at each sampling time \( k \):

\[
\begin{align*}
\min_{u(k+i|k); i=0,1,\ldots,M-1} J_p \left( \frac{x(k+j|k), u(k+j|k)}{j=1,\ldots,P} \right)
\end{align*}
\]

subject to constraints on the control input \( u(k+i|k), i = 0,1,\ldots,M-1 \) and also on the state \( x(k+i|k) \) and the output \( y(k+i|k), i = 0,1,\ldots,P \). \( J_p \) is the objective function to be minimized. The notation \( x(k+i|k) \) denotes the value of the variable \( x \) (which is either \( x, y \) or \( u \)) predicted at time \( k+i \) based on the measurements at time \( k \). \( P \) is the output or prediction horizon and \( M \) is the input or control horizon.

Implicit in the formulation is the assumption that a model (linear/nonlinear) is used to predict the system states and output. In the RHC framework, the first control move \( u(k|k) \) is implemented. At the next sampling time, new plant measurements are obtained and the optimization is solved again to recompute \( u \).

2.1 Feedback linearization and MPC

In this technique, the plant \( \dot{x} = f(x) + g(x)u, y = h(x) \) is linearized by the internal linearizing control loop [1] with \( u = \frac{1}{L_pL_f}h(x)+y-L_fh(x), -\bar{v} \leq v \leq \bar{v} \). The external MPC loop produces a control output \( u \). Note that a hard constraint on \( v \) is mapped into a state-dependent constraint on \( u \), leading to an implicit nonlinear programming problem in the external MPC loop. To overcome this problem, two possible approaches have been studied based on either approximating the state-dependent constraints by linear ones or using some iterative procedure to resolve the nonlinear constraints (see [5] for details). Conditions for global and local stability of the feasible FL-MPC scheme with the iterative procedure can be derived but will be skipped here for brevity.

Note that if there is a mismatch between the nonlinear model and the actual plant dynamics, the FL will not be exact and hence the FL-MPC method is likely to be sensitive to nonlinear plant model uncertainty.

2.2 Robust LMI-based MPC

The system under consideration is the following discrete linear time varying (LTV) system

\[ x(k+1) = A(k)x(k) + B(k)u(k), \]

\[ y(k) = Cx(k), \]

where \( \{A(k), B(k)\} \in \Omega \).

\( \Omega \) is the "uncertainty set" (see [2, 3] for details). The issue of approximating the nonlinear system (1) by a
LTV system is generally problem specific. For example, if the Jacobian $\frac{\partial f}{\partial x} \frac{\partial f}{\partial u}$ of (1) is known to lie in a polytope, then every trajectory $(x,u)$ of the original nonlinear system can be regarded as a trajectory of (3) for some LTV system in $\Omega$ (see [4]). Alternatively, in several situations, by imposing constraints on state variables, one can approximate nonlinear systems by uncertain LTV systems.

At each sampling time $k$, the robust performance objective to be minimized is as follows

$$\min \max_{i=0,1,\ldots,M-1} u(k+[i]k), \quad J_{\infty}(k), \quad (4)$$

where $J_{\infty}(k) = \sum_{i=0}^{\infty} (x(k+[i]k)^TQ_{1}x(k+[i]k) + u(k+[i]k)^TR_{u}(k+[i]k))$. An upper bound on this robust performance objective can be derived [2, 3] and the upper bound can be minimized with a constant state feedback control law $u(k+[i]k) = P_x(k+[i]k)$. Only the first computed input $u(k) = P_x(k)$ is implemented. At the next sampling time, the state $x(k+1)$ is measured and the optimization is repeated to recompute $F$. The problem of minimizing the upper bound can be reduced to a linear objective minimization problem subject to LMI constraints or equivalently, an eigenvalue problem (EVP)). Constraints on the control input $u$ and plant output $y$ can be incorporated as sufficient LMI conditions. Robust asymptotic stability of this feasible time-varying receding horizon state-feedback control strategy can be established using Lyapunov arguments (see [2, 3] for details).

### 3 Example

Consider the Van der Pol oscillator described by

$$\begin{cases}
 y' + \alpha(1-y^2)y + y = u
 \end{cases}$$

where $-1 \leq \alpha \leq 1$ is an uncertain constant with nominal value 1. Defining $x_1 = y, \quad x_2 = y, \quad x_3 = x_2, \quad x_4 = x_1$ we get the state-space equations

$$\begin{align*}
 x_1' = x_2, \\
 x_2' = -x_1 - \alpha(1-x_1^2)x_3 + x_4 \\
 x_3' = x_2, \\
 x_4' = -x_3.
\end{align*}$$

We want to robustly stabilize the system from an initial state $x_0$ to the origin using the bounded control $u, |u| \leq 10$. In the FL-MPC technique, the linearizing control input for the nominal system $\alpha = 1$ is given by $u = (1-x_1^2)x_3 + x_1 + v$ where $v$ is the output of the MPC algorithm. In order to apply the LMI-based MPC technique, we cover the uncertain parameter $\alpha$ and the nonlinearity $\alpha x_1^2$ with two uncertain nonlinear $\Delta$ blocks. This can be done by bounding the state $x_1$, i.e., $|x_1| \leq x_{\text{max}}$. Figure 2 shows the trajectory of state $x_2$ using the two control schemes, both for the nominal $\alpha = 1$ and the uncertain system $\alpha = -1$, starting from an initial state of $x(0) = [1 - 3]^T$. We see that the responses from both schemes are comparable for the nominal case. For the uncertain case, the feedback linearization is not exact and this leads to poor performance as seen in Figure 2. But the LMI-MPC scheme explicitly takes into account the uncertainty and hence gives better performance. It is conceivable that the FL-MPC scheme could give an unstable response for the uncertain case even when the nominal response is stable and well-behaved.

![Figure 2: System response. Top: LMI-MPC scheme; Bottom: FL-MPC scheme; solid: nominal case ($\alpha = 1$); dot-dash: uncertain case ($\alpha = -1$).](image)

### 4 Conclusions

The FL-MPC technique is applicable to a large class of feedback linearizable nonlinear systems. But as shown in this paper, its robustness properties are questionable. The LMI-MPC technique is applicable to a class of nonlinear systems which can be approximated by uncertain LTV systems. This technique comes with guarantees of robustness. As part of our future work, we would like to investigate the possibility of incorporating plant uncertainty in the FL-MPC scheme.

**Acknowledgment:** Partial financial support from the US Department of Energy and the National Science Foundation is gratefully acknowledged.

### References


