On Generalized Predictive Control: Two Alternative Formulations*

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Abstract—Two alternative solutions to the generalized predictive control (GPC) design problem are given for the case when the output horizon is larger than or equal to the plant order. First, using multirate models of the discrete-time case when the output horizon is larger than or equal to the predictive control (GPC) design problem are given for the recursions by the inversion of a lower triangular matrix. GPC can be easily derived from the transfer function plant, we present a state-space solution that does not require implementation. Second, it is shown, via simple algebraic manipulations, that the key prediction equation used for GPC can be easily derived from the transfer function coefficients effectively replacing the Diophantine equation recursions by the inversion of a lower triangular matrix.

1. Introduction
In the standard formulation of generalized predictive control (GPC) (see e.g. Clarke and Mohtadi (1987) and Clarke et al. (1987) and references therein) an input–output (algebraic) approach is taken and the controller derived from a transfer function representation of the plant. However, to determine the stability properties of the scheme, appeal is made to state-space models. On the other hand, standard solutions of quadratic minimization problems based on state-space models rely on state feedback and solution of a Riccati equation. It is our belief that the relationship between the latter solution and the solution obtained using the input–output approach is not totally understood. Our objective in this work is two-fold: first to provide a state–space formulation of the GPC that would be directly interpretable in terms of the derivations of Clarke and Mohtadi (1987) and Clarke et al. (1987). Second, to show that using simple algebraic manipulations the key prediction equation used in GPC can be easily derived without appealing to Diophantine equation recursions. The results presented in the paper apply to the case when \( N \geq n \), where \( N \) is the output horizon used in GPC and \( n \) is the plant order.

The key tool in the solution of the first problem is the use of a multirate model for the plant. In this model (Albertos, 1987) the single-input single-output plant is "lifted" to a multi-input multi-output representation where the inputs and outputs are now \( N \)th vectors composed of the shifted input and output sequences of the plant, respectively. A state-space representation for this new model is obtained with the state being updated every \( N \) units of time (\( T \) is the discrete time plant sampling period). This model is used by Albertos (1987) for the design of multirate controllers and in Francis and Georgiou (1987) and Khargonekar et al. (1985) to design robust periodic controllers. In this paper we use the model to obtain new relationships between the key prediction equation of GPC and the systems’ state–space realization. The main features of this formulation are: it does not require iterations of Diophantine or Riccati equations nor the use of observers for its implementation; and it extends without modification to the multi-input multi-output case.

The second alternative derivation of GPC is motivated by computational issues that arise in an adaptive implementation. Via simple algebraic manipulations, similar to the ones reported by Peterka (1984), it is shown that the key prediction equation (and therefore the optimal control sequence), can be computed directly from the coefficients of the plant transfer function with the inversion of only one lower triangular matrix.

In order to highlight the main ideas, and for ease of exposition, we treat here the basic version of GPC and present the derivation of the multirate model in its simplest possible form. We do not consider the presence of disturbances, nor the use of incremental models. Also we present the results for the case of a scalar plant with regulation objectives, however, as will become clear later, the results hold as well for multivariable plants with incremental models, disturbances and tracking objectives. The only restriction imposed on the GPC design is \( N \geq n \).

2. Input–output approach
In the standard derivation of the GPC for scalar plants a transfer function model is proposed

\[
A(q^{-1})y(kT) = q^{-1}B(q^{-1})u(kT), \quad k \in \mathbb{Z}^+, \quad (1a)
\]

where \( T \) is the sampling rate, \( A, B \) are polynomials in the delay operator, i.e. \( q^{-1}(kT) = q^{-j-1} \), of the form

\[
A(q^{-1}) = \sum_{i=0}^{n} a_i q^{-i}, \quad a_0 = 1, \quad B(q^{-1}) = \sum_{i=0}^{m} b_i q^{-i}. \quad (1b)
\]

Using the division algorithm of algebra we can define polynomials \( G_i(q^{-1}) \) and \( R_i(q^{-1}) \) as the unique solutions of

\[
B(q^{-1}) = G_i(q^{-1}) + q^{-j}R_i(q^{-1})B(q^{-1}), \quad j = 1, 2, \ldots, N \]

\[
A(q^{-1}) = G_i(q^{-1}) + q^{-j}R_i(q^{-1})A(q^{-1}), \quad j = 1, 2, \ldots, N \quad (2)
\]

where the order of the new polynomials are \( n_i = n + j - 1 \), \( n_{Ri} = n - 1 \) and \( N \) is the output horizon. Notice that

\[
B(q^{-1}) A(q^{-1}) = \sum_{i=0}^{n} h_i q^{i-j-1} \quad (3)
\]

\[
A(q^{-1}) = q^{-1}B(q^{-1}) - q^{-j-1}R_i(q^{-1})B(q^{-1}), \quad j = 1, 2, \ldots, N \]

\[
G_i(q^{-1}) = q^{-j-1}R_i(q^{-1})A(q^{-1}), \quad j = 1, 2, \ldots, N \quad (4)
\]

\[
R_i(q^{-1}) = 0, \quad j = N + 1, \ldots, N + n_i - n - 1 \quad (5)
\]

\[
B(q^{-1}) = q^{-1}A(q^{-1}) + q^{-j}R_i(q^{-1})B(q^{-1}), \quad j = 1, 2, \ldots, N \]

\[
A(q^{-1}) = q^{-1}B(q^{-1}) - q^{-j}R_i(q^{-1})A(q^{-1}), \quad j = 1, 2, \ldots, N \quad (6)
\]

\[
G_i(q^{-1}) = q^{-j}R_i(q^{-1})A(q^{-1}), \quad j = 1, 2, \ldots, N \quad (7)
\]

\[
R_i(q^{-1}) = 0, \quad j = N + 1, \ldots, N + n_i - n - 1 \quad (8)
\]

\[
B(q^{-1}) A(q^{-1}) = \sum_{i=0}^{n} h_i q^{i-j-1} \quad (9)
\]

\[
A(q^{-1}) = q^{-1}B(q^{-1}) - q^{-j-1}R_i(q^{-1})B(q^{-1}), \quad j = 1, 2, \ldots, N \]

\[
G_i(q^{-1}) = q^{-j-1}R_i(q^{-1})A(q^{-1}), \quad j = 1, 2, \ldots, N \quad (10)
\]

\[
R_i(q^{-1}) = 0, \quad j = N + 1, \ldots, N + n_i - n - 1 \quad (11)
\]
where $h_i$ are the impulse response coefficients (Markov parameters, see p. 70 of Kailath (1980)), of the plant. We will find it convenient to split $G(q^{-1})$ into

$$G(q^{-1}) = H(q^{-1}) + q^{-1}S(q^{-1}), \quad j = 1, 2, \ldots, N$$

(4)

where $n_j = j - 1$, $n_s = n - 1$.

Combining (1), (2) and (4) we derive

$$y[(k+j)T] = H(q^{-1})u[(k+j-1)T] + \theta_j\phi(kT),$$

(5)

where $\theta_j \in \mathbb{R}^{2n}$ contains the coefficients of $S_j(q^{-1})$ and $R_j(q^{-1})$, transposition is indicated by ' and

$$\phi(kT) \triangleq [u[(k-n)T], \ldots, u[(k-1)T]];$$

$$y[(k-n+1)T], \ldots, y[(kT)] \in \mathbb{R}^{2n}. \quad (6)$$

Notice that we have defined the terms of $\phi(kT)$ in a non-standard order.

By combining all the outputs into a vector $Y(kNT)$ one obtains the key prediction equation

$$Y(kNT) = HU(kNT) + \Theta \phi(kT) \quad (7)$$

where $H$ is a lower triangular Toeplitz matrix (p. 648 of Kailath (1980)) with the first column the first $N$ Markov parameters of the plant, i.e.

$$H = \begin{bmatrix} h_1 & 0 & 0 & \ldots & 0 \\ h_2 & h_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ h_N & h_{N-1} & \ldots & h_1 \end{bmatrix} \in \mathbb{R}^{N \times N} \quad (8)$$

and

$$Y(kNT) \triangleq [y[(k+1)T], y[(k+2)T], \ldots, y[(k+N)T]]^T \in \mathbb{R}^N \quad (9a)$$

$$U(kNT) \triangleq [u(kT), u[(k+1)T], \ldots, u[(k+N-1)T]]^T \in \mathbb{R}^N \quad (9b)$$

$$\Theta \triangleq [\theta_1^T, \theta_2^T, \ldots, \theta_N^T]^T \in \mathbb{R}^{N \times 2n} \quad (9c)$$

From the derivation above it is clear that the vectors $\theta_j$ are not independent (see Lemma 7.4.2 of Goodwin and Sin (1984)). To reduce the computational complexity in an adaptive implementation of the GPC a recursion of the Diophantine equation is usually carried out. One contribution of this paper is the derivation of two alternative expressions for $\Theta \phi(kT)$ the computation of which does not rely on recursions and can be easily computed from the plant transfer function coefficients.

The control sequence is chosen by GPC to minimize

$$J = \sum_{j=1}^{N} y[(k+j)T] + \lambda \sum_{j=0}^{N} u^2[(k+j)T], \quad \lambda > 0$$

leading to the periodic controller

$$U(kNT) = -(H^T H + \lambda I)^{-1} H^T \phi(kT). \quad (10)$$

In the following section we assume $N > n$ and derive the key prediction equation (7) proceeding from a multirate state-space model of the plant. Although the analysis is carried out for scalar plants, it applies as well, with trivial modifications, to the multivariable case.

3. Multirate state-space model formulation

Consider a minimal state-space realization of the plant

$$x[(k+1)T] = Ax(kT) + bu(kT), \quad A \in \mathbb{R}^{n \times n}, \ b \in \mathbb{R}^n \quad (12a)$$

$$y(kT) = c'x(kT), \quad c \in \mathbb{R}^s. \quad (12b)$$

We shall need an auxiliary state model which is motivated from the analysis of multirate systems presented by Albertos (1987). To this end, iterate (12a) to obtain

$$x[(k+i)T] = A'x(kT) + A'^{-1}bu(kT) + \cdots + bu[(k+i-1)T], \quad i \geq 1. \quad (12c)$$

For $i = N$ we have

$$x[(k+N)T] = A'^N x(kT) + C U(kNT) \quad (13)$$

where $U(kNT)$ is defined in (9b) and $C$ is the "$N$th controllability matrix" of (12)

$$C = [A^{N-1} b, A^{N-2} b, \ldots, b] \in \mathbb{R}^{s \times N}. \quad (14)$$

State equation (13) is the basic relationship used in this section. It may be interpreted as a discretization of the original continuous time plant with sampling period $NT$ and with controls applied every $T$ units of time, as shown in (12c), that is a multirate control. See Albertos (1987) for further discussions on multirate models. A similar equation can be obtained for the output vector (9a) as

$$Y(kNT) = Ox(kNT) + HU(kNT) \quad (15)$$

where $O$ and $H$ is as in (8) and $O$ is the "$N$th observability matrix" of (12) multiplied by $A$, that is

$$O \triangleq [A^c, (A^2)^c, \ldots, (A^N)^c]^T \in \mathbb{R}^{s \times N}. \quad (16a)$$

Notice that

$$h_i = c'A^{i-1}b, \quad i = 1, 2, \ldots, N. \quad (16b)$$

The state $x(kNT)$ can be easily reconstructed using (13) and (15) as

$$x(kNT) = A'^N O^* Y[(k-1)NT] + [C - A'^N O^* H] U[(k-1)NT] \quad (17)$$

where $(\cdot)^*$ denotes the pseudoinverse. Replacing (17) in (15) gives us the plant difference equation

$$Y(kNT) = O A'^N O^* Y[(k-1)NT] + O [C - A'^N O^* H] \times U[(k-1)NT] + H U(kNT). \quad (18)$$

The model (18) is an alternative, multi-input multi-output, representation of the plant (1). It may be derived in several ways: proceeding from a state-space model as above; via a lifting transformation of scalar into vector sequences as in Khargonekar et al. (1985) and Francis and Georgiou (1987) or as shown in the next section with simple iterations of the plant difference equation.

We are in a position to present the following result.

**Proposition 3.1.**

$$\Phi(kNT) = O A'^NO^* Y[(k-1)NT] + O [C - A'^NO^* H] U[(k-1)NT]. \quad (19)$$

**Proof.** Comparing (18) with the key prediction equation (7) gives us the proof.

Notice that (19) allows us to compute the optimal control (11) without appealing to recursions of Diophantine or Riccati equations nor the use of observers. In the next section it is shown how this expression reduces to a very simple form that can be computed directly from the coefficients of the polynomials $A(q^{-1})$ and $B(q^{-1})$ and the inversion of one lower triangular matrix.

4. Simplified derivation

The main result of this section, a simple derivation of the GPC for $N > n$ is presented in the form of a proposition below.
Proposition 4.1. Consider the plant (1) and the quadratic criterion (10). Assume $N > n$. Then the sequence of optimal controls is given by

$$U(kNT) = -(B_i^t L + L_{ij}^{-1} B_i^t L_i) \times (B_i^t U((k-1)NT) + A_i Y((k-1)NT))$$  \hspace{1cm} (20a)$$

$$L_i = [(1-A_i^t)(1-A_i^t)]^{-1}$$  \hspace{1cm} (20b)$$

where $A_i, B_i, B_2, A_2 \in \mathbb{R}^{N \times N}$ are Toeplitz matrices defined as follows:

$A_i$: lower triangular with first column $[0, -a_1, \ldots, -a_n, 0, \ldots, 0]^t$

$B_i$: lower triangular with first column $[b_0, b_1, \ldots, b_n, 0, \ldots, 0]^t$

$B_2$: upper triangular with first row $[0, \ldots, 0, b_n, b_{n-1}, \ldots, b_1]$

$A_2$: upper triangular with first row $[0, \ldots, 0, a_n, \ldots, -a_1]$

(21)

Proof. From (1), (9a), (9b) and the matrices defined by (21) it is easy to show that

$$Y(kNT) = A_1 Y(kNT) + B_1 U(kNT) + B_2 U((k-1)NT) + A_2 Y((k-1)NT).$$  \hspace{1cm} (22)$$

Comparing (22) with the key prediction equation (7) and noting that $L_i - A_i$ is nonsingular, we get the identities

$$H = (I - A_i)^{-1} B_1$$  \hspace{1cm} (23a)$$

$$\Theta \phi(kNT) = (I - A_i)^{-1} (B_i^t U((k-1)NT) + A_2 Y((k-1)NT)).$$  \hspace{1cm} (23b)$$

The proof is completed replacing (23) in (11).

Equations (22) and (18) are proposed by Albertos (1987) to model multirate sampled data systems. While (23a) is a well-known identity (see e.g. p. 102 of Kailath (1980)), the following relations obtained from (22) and (18) seem to be new

$$O^t A O = A_2, \quad O^t C - A O^t H = B_2.$$  \hspace{1cm} (24)$$

Let us briefly summarize below the steps for an adaptive implementation of GPC using the simplified derivation (20a).

Step 1. Choose $\lambda, N \geq n$ and (possibly) $N_c$—control horizon. Set $k = 0$, initialize the identification algorithm and vectors $Y(kNT), U(kNT), (9)$.  

Step 2. Set $k = k + 1$ and compute the estimates of $a_i, b_i$ using the regression model (5), $j = 1, \ldots, n$ is

$$y_j[(k+1)T] = b_0 w(kT) + \theta_1 \phi(kT).$$

Step 3. Replace the estimates in $A_1, A_2, B_1, B_2$ (21) and calculate $U(kNT)$ from (20a) (if $N_c \neq 1$ truncate $H$ accordingly).

Step 4. Shift the elements in $\phi(kT), Y(kNT), U(kNT)$. Go to Step 2.

5. Conclusions

Two alternative derivations of the GPC have been presented in the paper. The first derivation relies on a multirate state-space model formulation of the plant. This has allowed us to derive an expression for the optimal control the computation of which does not require recursions or the use of observers. Another advantage of this approach is that it applies to the multivariable case, a task which is considerably more involved when using recursions of polynomial matrices.

The second form of the GPC is obtained via a simple iteration of the plant difference equations. It leads to an expression for the key prediction equation that can be computed directly from knowledge of the parameters of the plant transfer function and requires only, in addition to the standard computations, the inversion of a simple lower triangular matrix.

In closing, it is interesting to note that although there exists a lot of interest in the theoretical literature for the use of periodic compensators to improve robustness (Francis and Georgiou, 1987; Khargonekar et al., 1985), practitioners suggest, as in GPC, receding horizon policies to avoid the periodicity!

References


