Cutoff and leakage properties of bi-soliton and its existent parameter range

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ABSTRACT

A periodically stationary pulse propagating in a dispersion-managed optical fiber transmission line was found with a numerical averaging method and named a dispersion-managed soliton. On the other hand, it has been pointed out that such a pulse naturally has energy leakage property. In addition, it has been reported that the existent parameter range of bi-soliton which is a periodically stationary pair of adjacent pulses is limited. In this paper, we investigate the cutoff and leakage properties of dispersion-managed solitons including uni- and bi-soliton, and show then the limiting factor of their existent parameter range s.

Keywords: optical fiber communications, optical soliton, dispersion-management

1. INTRODUCTION

Dispersion-managed(DM) line in which the fiber’s dispersion is intentionally changed along the line periodically is an essential technique to achieve a long-haul and large-capacity transmission. A periodically stationary pulse propagating in such a DM line was found by a numerical averaging method and named a DM soliton (DMS)\(^1\). On the other hand, it has been pointed out that such a pulse naturally has energy leakage property due to the parametric resonance between the pulse and the dispersive wave\(^2,3,4\). The leakage property limits the existent parameter range in which a DMS propagates as a periodically stationary pulse. A periodically stationary pair of adjacent pulses propagating in a DM line has been recently found by the numerical averaging method and named a bi-soliton\(^5,6\). The existence of bi-soliton has been also confirmed by a multiple-scale method\(^7\) and averaged variational method\(^8\). It has been pointed out that the existent parameter range of bi-soliton is also limited\(^5,6,7\). In this paper, we investigate the energy leakage property of uni-soliton i.e., conventional DMS and bi-soliton and the bi-soliton’s cutoff property by which a bi-soliton loses the periodically stationary property. These properties limit the DMS’s existent parameter range.

2. GENERAL PROPERTY OF A PULSE PROPAGATING IN A DM LINE

The optical pulse propagation in a DM system in which the dispersion, nonlinearity and gain (or loss) coefficients are periodic functions of \(Z\) with the identical period of \(T\) can be described by\(^6\)
\[
i \frac{\partial U}{\partial Z} - \frac{b(Z)}{2} \frac{\partial^2 U}{\partial T^2} + \|U\|^2 U = 0
\]
where \(U(Z,T)\) represents the normalized complex envelope of electric field. \(b(Z)\) is fiber’s effective dispersion. \(Z\) is the transmission distance measured with the accumulated nonlinearity and \(T\) is the retarded time coordinate moving with the group velocity. These are normalized with a period of dispersion management and minimum pulse width in a period, respectively\(^6\).

We assume that an optical pulse propagating in a DM system has the following form including the linear chirp\(^9\),
\[
U(Z,T) = A(Z)\nu(Z,\tau) \exp\left[i \frac{C(Z)}{2} \tau^2 \right]
\]
where \(\tau = p(Z)T\). \(A(Z), p(Z)\) and \(C(Z)\) are real functions of \(Z\) and represent the variations of pulse’s
amplitude, reciprocal of pulse width, and linear chirp, respectively. Substituting Eq. (2) into (1), we obtain

\[
\begin{align*}
&i \frac{\partial v}{\partial Z} + \frac{1}{p} \left( \frac{dp}{dZ} - b p^3 C \right) \frac{\partial v}{\partial \tau} - \frac{b p^2}{2} \frac{\partial^2 v}{\partial \tau^2} + A^2 \left| v \right|^2 v \\
&\quad + \left\{ i \left( \frac{dA}{dZ} - \frac{b}{2} A p^3 C \right) - \frac{\tau^2}{2} \left( \frac{dC}{dZ} + 2 \frac{dp}{p \, dZ} - b p^2 C^2 \right) \right\} v = 0.
\end{align*}
\]

(3)

When \( A(Z), p(Z) \) and \( C(Z) \) satisfy the following ordinary differential equations,

\[
\begin{align*}
&\frac{dA}{dZ} = \frac{b(Z)}{2} - A p^3 C, \\
&\frac{dp}{dZ} = b(Z) p^3 C, \\
&\frac{dC}{dZ} = -b(Z) p^2 C^2 + \kappa(Z),
\end{align*}
\]

(4) \hspace{1cm} (5) \hspace{1cm} (6)

Eq. (3) can be reduced to

\[
\begin{align*}
&i \frac{\partial v}{\partial Z} - \frac{b p^2}{2} \frac{\partial^2 v}{\partial \tau^2} + A^2 \left| v \right|^2 v - \frac{\kappa}{2} \tau^2 v = 0.
\end{align*}
\]

(7)

In Eq. (7), it has been pointed out that the variation of \( \left| v(Z, \tau) \right| \) along \( Z \) is much less than those of the coefficients, \( b(Z) p^2(Z), A^2(Z), \) and \( \kappa(Z) \). Therefore we consider the following averaged equation of Eq. (7),

\[
\begin{align*}
&i \frac{\partial \bar{V}}{\partial Z} - \frac{B_0}{2} \frac{\partial^2 \bar{V}}{\partial \tau^2} + A_0 \left| \bar{V} \right|^2 \bar{V} - \frac{K_0}{2} \tau^2 \bar{V} = 0,
\end{align*}
\]

(8)

where \( B_0 = \left\langle b(Z) p^2(Z) \right\rangle, A_0 = \left\langle A^2(Z) \right\rangle, \) and \( K_0 = \left\langle \kappa(Z) \right\rangle \) represent the effective dispersion, the effective nonlinearity, and the effective chirp coefficients, respectively. \( \langle \cdot \rangle \) represents the averaged value over a DM period.

\( V(Z, \tau) \) represents the averaged pulse's core. Assuming \( \left| V(Z, \tau) \right| \) does not change along \( Z \), we set \( V(Z, \tau) = F(\tau) \exp(\tilde{\lambda}_0 Z) \) with a real function of \( F(\tau) \) and substitute it into Eq. (8), we get

\[
- \frac{B_0}{2} \frac{d^2 F}{d\tau^2} - \left( -A_0 F^2 + \frac{K_0}{2} \tau^2 \right) F = \tilde{\lambda}_0 F.
\]

(9)

Equations (8) and (9) have the same forms as the Schroedinger equation in the quantum mechanics and their potential can be expressed as

\[
\Phi(\tau) = -A_0 F^2(\tau) + \frac{K_0}{2} \tau^2.
\]

(10)

While the first term in the right hand side of Eq. (10) represents the potential induced by the nonlinearity which is proportional to the pulse's intensity, the second term represents the potential induced by the pulse's chirp. The pulse shape of \( F(\tau) \) is determined by the potential of \( \Phi(\tau) \) and has quite different form dependent on the sign of \( K_0 \). For \( K_0 > 0 \), the both potentials confine pulse's energy in the vicinity of \( \tau = 0 \). On the other hand, for \( K_0 < 0 \), while the nonlinearity induced potential confines the pulse's energy in the vicinity of \( \tau = 0 \), the pulse's energy leaks towards \( |\tau| \rightarrow \infty \) due to the decreasing chirp-induced potential for large \( |\tau| \). Figure 1 shows typical potentials of Eq. (10) for \( K_0 \leq 0 \).
Next, let us prove that $K_0$ is zero or negative in any DM line. We set $v(Z,\tau) = f(\tau)\exp[\theta(Z)]$ in Eq.(2) and apply the Lagrange’s variational method to Eq.(1), we then have

\[
\frac{dC}{dZ} = -b(Z)p^2\left(C^2 + \frac{I_D}{I_C}\right) - \frac{I_N A^2}{2 I_C}.
\]

Using Eqs. (1) and (2), we have a constant

\[
E_0 = \int_{0}^{\infty} |U(Z,T)|^2 dT = \frac{A^2(Z)}{p(Z)} I_L.
\]

In Eqs.(11) and (12),

\[
I_L = \int_{0}^{\infty} f^2 d\tau, \quad I_D = \int_{0}^{\infty} \left(\frac{df}{d\tau}\right)^2 d\tau, \quad I_C = \int_{0}^{\infty} (\tau f)^2 d\tau, \quad I_N = \int_{0}^{\infty} f^4 d\tau.
\]

Comparing Eqs.(6) and (11), we notice

\[
\kappa(Z) = -\alpha_1 b(Z)p^2(Z) - \alpha_2 p(Z),
\]

where

\[
\alpha_1 = \frac{I_D}{I_C}, \quad \alpha_2 = \frac{I_N E_0}{2 I_L I_C},
\]

and the both are positive constants. Using Eqs. (11) and (15), we obtain

\[
\frac{\alpha_1}{\alpha_1 + C^2(Z)} \frac{dC}{dZ} = -\alpha_1 b(Z)p^2(Z) - \frac{\alpha_2}{\alpha_1 + C^2(Z)} p(Z).
\]

We then calculate the average of the left hand side of Eq.(16) over a DM period for a periodically stationary pulse in which $p(Z)$ and $C(Z)$ are periodic functions.

\[
\left\langle \frac{\alpha_1}{\alpha_1 + C^2} \frac{dC}{dZ} \right\rangle = \int_{0}^{1} \alpha_1 \left(\frac{dC}{dZ} - \frac{\alpha_1}{\alpha_1 + C^2} dC\right) d\tau = 0
\]

The average of the right hand side of Eq.(16) satisfies

\[
\left\langle \alpha_1 b(Z)p^2(Z) \right\rangle = -\left(\frac{\alpha_2}{\alpha_1 + C^2(Z)} p(Z)\right).
\]

Therefore the average of the right hand side of Eq.(14) is
\[ K_0 = -\left\langle \alpha b(Z) p^2(Z) \right\rangle - \left\langle \alpha^2 p(Z) \right\rangle = \frac{\alpha^2 C^2(Z)}{\alpha + C^2(Z)} p(Z) \leq 0. \] (19)

\( K_0 \leq 0 \) has been finally proved.

For \( K_0 < 0 \), the oscillating component propagating towards \( t \to \infty \) appears at the pulse tails and is called dispersive wave. Since the dispersive wave leaks from the nonlinearity induced potential arising from the tunnel effect, pulse’s energy is gradually decreasing during the propagation.

As we have shown above, any pulse propagating in a DM line is leaky. Now then, how large energy does the pulse leak during the propagation? We know that the guiding center soliton which exists in a DM line in which the variation of dispersion is not so large is accompanied with negligible energy leakage. In the following sections, we investigate the energy leakage property of uni- and bi-soliton using numerical simulations.

Only for simplicity, we consider a symmetric 2-steps DM system in which \( b(Z) = b_1 (,< 0) \) (anomalous dispersion) for \( \left| Z - n \right| < l_1 / 2 \) and \( b(Z) = b_2 (> 0) \) (normal dispersion) for \( l_1 / 2 < \left| Z - n \right| \) where \( n \) is an integer. We introduce the following three system parameters, the accumulated dispersion \( B \equiv b_1 l_1 + b_2 (1 - l_1) \), the map strength \( S \equiv -b_1 l_1 + b_2 (1 - l_1) \), and the ratio of accumulated nonlinearity in the fiber of \( b(Z) = b_1 \) to the total accumulated nonlinerity \( R \equiv l_1 \), which completely characterize a uni-soliton.

### 3. ENERGY LEAKAGE PROPERTY OF UNI-SOLITON AND ITS EXISTENT PARAMETER RANGE

Firstly, we calculated the energy \( E_0 \) of a single quasi-periodically stationary pulse using a numerical averaging method\(^9\). The pulse energy is measured when the full width at half maximum is 1. Figure 2 shows the variation of pulse energy against the accumulated dispersion \( B \) for \( R = 0.5 \) and \( S = 0, 1, 2, 3 \). While the filled symbol represents the uni-soliton’s energy, the open one the leaky pulse’s energy obtained by the averaging method. The dashed line represents the pulse energy obtained by the variational method in which a gaussian pulse having linear chirp is used as a test function.\(^9\) The dotted line for \( S = 0 \) represents the soliton’s energy propagating in a constant dispersion line for \( b(Z) = B \) (constant). We here introduce the energy leakage factor \( \eta \) shown in Fig. 3 to distinguish the uni-soliton and the leaky pulse. The leakage factor \( \eta \) is calculated with the following way. The averaged pulse which energy is \( E_0 \) and width is 1 propagats in a DM line without averaging. We measure the pulse energy \( E' \) after propagation over 1000 periods and calculate \( \Delta E \equiv E_0 - E' \) which represents the energy of dispersive wave emitted from the pulse. We define the energy leakage factor \( \eta \) with the ratio of \( \Delta E \) and \( E_0 \) as \( \eta \equiv \Delta E / E_0 = 1 - E' / E_0 \). In Fig.2, we considered that the pulse is periodically stationary and can be called uni-soliton when \( \eta \leq 10^{-7} \) and is a leaky pulse when \( \eta > 10^{-7} \). Figure 2 shows that the upper limit of \( |B| \) where uni-soliton can exist is larger for larger \( S \).

![Figure 2: Variation of pulse energy \( E_0 \) against the accumulated dispersion \( B \). While the filled symbol represents the uni-soliton, the open one leaky pulse.](image-url)
4. ENERGY LEAKAGE AND CUTOFF PROPERTIES OF BI-SOLITON AND ITS EXISTENT PARAMETER RANGE

Firstly, we calculated the energy $E_0$ of a quasi-periodically stationary pair of adjacent pulses using the numerical averaging method. The pulse energy is measured when the full width at half maximum of a pulse is 1. Figure 4 shows the variation of pulse energy and pulse-to-pulse spacing against the accumulated dispersion $B$ for $R = 0.5$ and $S = 2$. While the filled symbol represents the uni- or bi-soliton’s energy, the open one the leaky pulse’s or pulse pair’s energy obtained by the averaging method. The cross represents the pulse-to-pulse spacing of the bi-soliton. The dashed line represents the energy of the single pulse obtained by the variational method. The dotted line represents the double energy of the single pulse. As the same as the above section, the energy leakage factor $\eta$ shown in Fig. 5 is used to distinguish the bi-soliton and the leaky pulse pair. In Fig.4, we considered that the pulse pair is periodically stationary and can be called bi-soliton when $\eta \leq 10^{-7}$ and is a leaky pulse pair when $\eta > 10^{-7}$. In this case, the boundary between bi-soliton and leaky pulse pair locates at $B \approx -0.135$. The dotted and solid lines in Fig. 6 respectively show the typical temporal waveforms of a bi-soliton and leaky pulse pairs obtained by the averaging method. The dispersive wave propagating to $|T| \rightarrow \infty$ can be clearly seen for $B = -0.3$ and $0.4$. Figure 4 shows that the energy of bi-soliton is less than that of infinitely apart two uni-solitons. This indicates that the bi-soliton is stabilized by the bonding energy. It is similar to the hydrogen molecule which is more stable than two independent hydrogen atoms. We also noticed that the pulse-to-pulse spacing of bi-soliton is almost linearly increases with the accumulated dispersion $|B|$. 

![Fig. 3: Variation of the energy leakage factor $\eta$ of a single pulse against the accumulated dispersion $B$.](image)

![Fig. 4: Variation of pulse energy $E_0$ and pulse-to-pulse spacing $T_s$ against the accumulated dispersion $B$. While the filled symbol represents the uni- or bi-soliton, the open one the leaky pulse or pulse pair. The cross represents the pulse-to-pulse spacing of the bi-soliton.](image)
Fig. 5: Variation of the energy leakage factor $\eta$ of a pair of adjacent pulses against the accumulated dispersion $B$.

Figure 6: Comparison of typical temporal waveforms of a bi-soliton (dotted line) and leaky pulse pairs (solid lines) obtained by the averaging method.

Figure 7 shows the variation of pulse energy (filled circle) and pulse-to-pulse spacing (cross) of bi-soliton against the map strength $S$ for $R = 0.5$ and $B = -0.1$. The filled square and dashed line represent the energy of uni-soliton obtained by the averaging and variational methods, respectively. The dotted line represents the double energy of uni-soliton. We didn’t find any leaky pulse pair for parameters in Fig. 7. Since the energy leakage factor $\eta < 10^{-8}$ in the range of $1.45 < S < 2.66$, bi-soliton exists. The energy of bi-soliton asymptotically approaches to that of infinitely apart two uni-solitons for smaller $S$. This observation indicates that the bonding energy of bi-soliton weakens for smaller $S$. Since bi-soliton does not exist for $S < 1.45$ and the pulse-to-pulse spacing of bi-soliton increases rapidly for smaller $S$, we may predict that the launched pair of adjacent pulses into a DM line separate away each other and finally become two independent uni-solitons. On the other hand, the launched pair of adjacent pulses into a DM line are attracted each other and finally collide for $S > 2.66$.

As we have shown in Figs. 4 and 7, the existent parameter range of bi-soliton is limited not only by the energy leakage property which is general for a pulse propagating in a DM line but also by the cutoff property by which the existence of a pulse pair with a fixed pulse-to-pulse spacing is not permitted.
5. CONCLUSIONS

In this paper, we have investigated the existent parameter ranges of uni-soliton and bi-soliton which are a periodically stationary pulse and pair of adjacent pulses propagating in a DM line using numerical averaging method. In any numerical calculation, we should set a suitable criterion to distinguish between DM soliton and leaky pulse (pair). The criterion may however be subjective. Though one notices that the energy leakage occurs even in the parameter range in which \( \eta < 10^{-7} \) from the energy leakage factor shown in Figs. 3 and 5, we considered that the DM soliton exists in Figs. 2 and 4. Therefore DM soliton essentially has the energy leakage property. In addition to the energy leakage property, the existent parameter range of bi-soliton is limited by the cutoff property. At the outer sides of cutoff points, the launched pair of adjacent pulses separate away due to the decrease of bonding energy or are attracted each other due to the over-increase of attractive forces.

REFERENCES


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