ABSTRACT

A split step transfer matrix formulation for non-paraxial beam propagation method is presented in this paper. We present a general formulation that allows the splitting of the propagation operator into two distinct operators, i.e., one representing propagation in a uniform medium, and the other, representing the guiding effect of the optical waveguiding structure. The two operators are defined in the form of matrices that are multiplied with the input field to obtain the final, output field after the desired propagation distance. Thus, the formulation requires simple repeated matrix multiplication for beam propagation even through complicated waveguides, and the methodology is general enough to be applicable in different propagation schemes. We present here the implementation in the finite difference method along with some key results that show the accuracy of the formulation, and present some details of implementation in the collocation method for completeness.

Keywords: Wide angle beam propagation, symmetrized splitting, finite difference method.

1. INTRODUCTION

Recently several schemes have been suggested for wide-angle and bi-directional beam propagation through guided-wave devices. In general, this non-paraxial propagation would involve solving directly the wave equation, which contains a second order partial derivative with \( z \) (the general direction of propagation) as against the first order partial derivative in the paraxial wave equation. All the methods for non-paraxial beam propagation discussed in the literature approach this problem iteratively, in which a numerical effort equivalent to solving the paraxial equation several times is involved. The actual number of iterations depends on the desired accuracy and the obliquity of the beam. Many of these methods neglect the backward propagating components and solve the one-way wave equation; but even the methods that deal with bi-directional propagation employ special techniques either to suppress or to model evanescent modes, which are a source of instability in these methods. In all these methods, the square root of the propagation operator involved in the wave equation is approximated in various ways. One of the approximations used is based on the Padé approximants. Recently we have proposed a new method based on symmetrized splitting of the operator for non-paraxial propagation. In this paper, we briefly discuss this method and its implementation in the finite difference scheme.

2. OUTLINE OF METHOD

The two-dimensional scalar wave equation is given by

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2 n^2(x, z) \psi(x, z) = 0
\]

Equation (1) can be rewritten as

\[
\frac{\partial F}{\partial z} = H(z) F(z),
\]

where

\[
F(z) = \begin{bmatrix} \psi \\ \frac{\partial \psi}{\partial x} \end{bmatrix}, \quad H(z) = \begin{bmatrix} 0 & 1 \\ -\nabla^2 - k_0^2 n^2 & 0 \end{bmatrix}.
\]

The operator \( H \) can be written as a sum of two operators, one representing the propagation through a uniform medium of index, say \( n_r \), and the other representing the effect of the index variation of the guiding structure; thus,
\[ H(z) = H_1 + H_2(z) \]

\[
= \begin{bmatrix}
0 & 0 \\
-\nabla r^2 - k_0^2 n_r^2 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
k_0^2 (n_r^2 - n_t^2) & 0
\end{bmatrix}
\]

A formal solution of the above equation after symmetrized splitting of operators can be written as

\[ F(z + \Delta z) = P Q(z) F(z) + \left( \left( \Delta z \right)^3 \right) \]

\[ P = e^{z H_1 \Delta z}, \quad Q(z) = e^{H_2 \Delta z} . \]

where \( P \) and \( Q(z) \) represent the propagation in uniform medium of index \( n_r \) and effect of the refractive index variation of the guiding structure, respectively. The concept of splitting of operators is independent of the scheme modeling propagation. The evaluation of \( Q(z) \) is exact; however the evaluation of \( P \) introduces a discretization error in \( \Delta z \) as well as \( \Delta x \).

### 2.1 Implementation in the Collocation Method

In the collocation method, the wave equation is converted to a matrix ordinary differential equation

\[
\frac{d^2 ?}{dz^2} + \left[ S_0 + k_0^2 n_r^2 I + R(z) \right] ?(z) = 0
\]

using the representation of the field \( \psi(x, z) \) as a linear combination of a set of orthogonal basis functions, \( \phi_n(x) \):

\[ \psi(x, z) = \sum_{n=1}^{N} c_n(z) \phi_n(x) \]

where \( c_n(z) \) are the expansion coefficients, \( n \) is the order of the basis functions and \( N \) is the number of basis functions used in the expansion. Further,

\[
? (z) = \begin{bmatrix}
\psi(x_1, z) \\
\psi(x_2, z) \\
\vdots \\
\psi(x_N, z)
\end{bmatrix}, \quad R(z) = k_0^2 \begin{bmatrix}
\Delta n^2(x_1, z) & 0 & 0 \\
0 & \Delta n^2(x_2, z) & 0 \\
0 & 0 & \Delta n^2(x_N, z)
\end{bmatrix}
\]

where \( \Delta n^2(x_m, z) = n^2(x_m, z) - n_r^2 \), \( m = 1, 2, \ldots, N \), and \( S_0 \) is a constant known matrix defined by the basis functions. The choice of \( \phi_n(x) \) depends on the boundary conditions and the symmetry of the guiding structure. The coefficients of expansion, \( c_n(z) \), are unknown and represent the variation of the field with \( z \). The formal solution of Eq. (7) can be written as given in Eq. (5), with the operators \( P \) and \( Q \), and the field function \( F \) now being block matrices.

\[
F(z) = \begin{bmatrix}
? \\
\frac{d^2?}{dz^2}
\end{bmatrix}, \quad P = \exp \left\{ \begin{bmatrix}
\Delta z & 0 & 0 \\
\frac{-1}{2} (S_0 + k_0^2 n_r^2 I) & I
\end{bmatrix} \right\}, \quad Q(z) = \begin{bmatrix}
I & 0 \\
-R(z) & I
\end{bmatrix}
\]

where \( I \) and \( 0 \) the unit and null matrices, respectively. The results obtained using this implementation are better than existing methods and are discussed in Ref. 15.

### 2.2 Implementation in the finite difference method

Following the formulation in Section 2, solution of the wave equation is given by Eq. 5, where the operators \( P \) and \( Q(z) \) represent the propagation in uniform space and effect of the refractive index variation of the guiding structure.
respectively. Thus, evaluation of \( P \) corresponds to the solution of the wave equation with a constant refractive index, \( n_r \). Evaluation of \( P \) is nothing but the solution of:

\[
\begin{align*}
\frac{\partial^2 \psi}{\partial x^2} &= \frac{1}{\Delta x^2} \left( \frac{\Delta c^2}{\Delta z} + \frac{\Delta z^2}{6} \right) \psi + O[(\Delta z)^6] \\
\frac{\partial^2 \psi}{\partial z^2} &= -k_0^2 n_r^2 \psi \\
\frac{\partial \psi}{\partial z} &= \frac{1}{\sqrt{k_0^2 n_r^2}} \sin(\sqrt{k_0^2 n_r^2} \Delta z) \psi + O[(\Delta z)^5]
\end{align*}
\] (11)

The solution of Eq. 11 is of the form

\[
\begin{bmatrix}
\psi(z + \Delta z) \\
\psi'(z + \Delta z)
\end{bmatrix} =
\begin{bmatrix}
\cos(\sqrt{k_0^2 n_r^2} \Delta z) & \frac{1}{\sqrt{k_0^2 n_r^2}} \sin(\sqrt{k_0^2 n_r^2} \Delta z) \\
-\frac{\sqrt{k_0^2 n_r^2}}{\cos(\sqrt{k_0^2 n_r^2} \Delta z)} & \cos(\sqrt{k_0^2 n_r^2} \Delta z)
\end{bmatrix}
\begin{bmatrix}
\psi(z) \\
\psi'(z)
\end{bmatrix}
\] (12)

where the operator \( S \) now is a finite-difference (FD) representation of \( \frac{\partial^2}{\partial x^2} + k_0^2 n_r^2 \). In Sec. 2.1, we described an eigenvalue decomposition method for the evaluation of Eq. 5. Here we obtain an explicit, transfer matrix form for \( P \) where the accuracy can be increased to arbitrary order in \( \Delta x \), while retaining the same computational efficiency.

By expanding the \( \cos \) and \( \sin \) terms in Eq. 12, we can determine the order of accuracy, limited only by the error caused by splitting in \( \Delta z \), while, by choosing higher order terms to evaluate \( \partial^2 / \partial x^2 \), we decrease the error in \( \Delta x \) to a desired level. The matrix size remains the same. Thus, using:

\[
\begin{align*}
\cos(\sqrt{k_0^2 n_r^2} \Delta z) &= 1 - \frac{(\sqrt{k_0^2 n_r^2} \Delta z)^2}{2!} + \frac{(\sqrt{k_0^2 n_r^2} \Delta z)^4}{4!} - \cdots + O[(\Delta z)^6] \\
\sin(\sqrt{k_0^2 n_r^2} \Delta z) &= \frac{\sqrt{k_0^2 n_r^2} \Delta z}{3!} - \frac{(\sqrt{k_0^2 n_r^2} \Delta z)^3}{5!} - \cdots + O[(\Delta z)^7]
\end{align*}
\] (13)

Eq. 12 can be written as

\[
\begin{bmatrix}
\psi(z + \Delta z) \\
\psi'(z + \Delta z)
\end{bmatrix} =
\begin{bmatrix}
I - \frac{\Delta z^2}{2} S + \frac{\Delta z^4}{24} S^3 & \Delta c I - \frac{\Delta z^3}{6} S + \frac{\Delta z^5}{120} S^3 \\
-\Delta c S + \frac{\Delta z^3}{6} S^2 - \frac{\Delta z^5}{120} S^3 & I - \frac{\Delta z^2}{2} S + \frac{\Delta z^4}{24} S^3
\end{bmatrix}
\begin{bmatrix}
\psi(z) \\
\psi'(z)
\end{bmatrix} + O[(\Delta z)^5]
\] (14)

Equation 14 is a transfer matrix split step finite difference solution to the wave equation in a uniform medium.

The accuracy of propagation in the waveguide is determined by the error caused due to symmetrized splitting, defined in Eq. 5, and by the accuracy of the uniform propagation operator, \( P \). Equation 14 is an FD representation of this operator. By taking a larger number of terms in the expansion of the cosine and sine series, we increase the accuracy in \( \Delta z \), which is already limited by splitting. However, the accuracy in \( \Delta x \) is not limited by splitting and can be increased indefinitely. By choosing higher order terms to evaluate \( \partial^2 / \partial x^2 \), we decrease the error in \( \Delta x \) to a desired level without any significant increase in the propagation time. The term \( \partial^2 / \partial x^2 \) can be written in the form:

\[
\frac{\partial^2}{\partial x^2} = \frac{1}{\Delta x^2} \left( \delta_x^2 + \frac{1}{12} \delta_x^4 + \frac{1}{90} \delta_x^6 - \frac{1}{560} \delta_x^8 + \cdots \right)
\] (15)

where \( \delta_x^p \psi_p = \psi_{p+1} - 2\psi_p + \psi_{p-1} \), and the \( \delta_x^2 \) operator can be represented by a tri-diagonal matrix. Use of the first term in the series given by Eq. 15, corresponds to the Crank-Nicholson scheme and the first two terms, to the Generalised Douglas scheme, both of which are standard finite difference schemes. The lowest power in the above series expansion is 1, corresponding to the 3 point central difference scheme with an error of the order of \( \Delta x^2 \) for the Crank-Nicholson scheme. As the number of terms in the series expansion is increased, the matrix becomes denser, and the accuracy in \( \Delta x \) increases indefinitely. Increasing the number of terms in the expansion is numerically similar to increasing the Padé order in the conventional wide-angle methods, without having to adopt an iterative multi-step procedure that the Padé analysis requires. Also, the increase in matrix bandwidth does not alter the computation effort or efficiency, as the number of matrix multiplication required for propagation does not depend on the bandwidth of \( S \).

Physically, increasing the number of terms in the series in Eq. 15, corresponds to an increase in the number of points in
the $x$ direction which are used in approximating $\partial^2 / \partial x^2$, leading to a better and better representation of the derivative with respect to $x$, which yields a more accurate representation of $P$. Thus, the method we have presented, allows for increase in accuracy in the transverse domain by making an approximation for $\partial^2 / \partial x^2$ that can be improved in accuracy to arbitrary order, without having to adopt an iterative, multi-step procedure required in the conventional Padé approximant based methods, while the error in the propagation direction is limited only by splitting.

3. RESULTS AND DISCUSSION

We consider the example of the propagation of the fundamental mode through a tilted graded-index waveguide, with index profile given by $n^2(x) = n_s^2 + 2n_s \Delta n \text{sech}^2(2x/w)$, $n_s = 2.1455$, $\Delta n = 0.003$, $w = 5 \mu m$ and $\lambda = 1.3 \mu m$ for a propagation distance of $100 \mu m$ with a propagation step size of $0.05 \mu m$. As a measure of accuracy, we computed an error ($ERR$), which includes the effects of both the dissipation in power as well as the loss of shape of the propagating mode:

$$ERR = 1 - \left( \frac{\left| \int \psi_{exact} \psi_{calc}^* dx \right|^2}{\left( \int \psi_{calc}^* dx \right) \left( \int \psi_{exact}^* dx \right)} \right)$$

(16)

where $\psi_{inp}, \psi_{calc}$ and $\psi_{exact}$ are the input, the propagated and the exact fields, respectively. Figure 1 shows the variation in accuracy as higher order terms in the series of Eq. 14 are considered. It can be clearly seen that with increase in the number of terms, the accuracy in propagation increases substantially. This improvement in accuracy is not accompanied by an increase in computation time. This fact is illustrated by Fig.2, which shows separately the time required ($t_S$) for the one time computation of $S$ and the time required ($t_{PROP}$) for propagation $s$. The figure shows that although as expected there is an increase in $t_S$ with the increase in order of exponent (in Eq. 15), but the time required for

![Fig. 1 Error in overlap integral with order of exponent of the largest term in the series expansion, for propagation at 50° waveguide tilt angle.](image)
propagation, $t_{\text{prop}}$, is independent of this order. To compare the method with other wide-angle methods reported in the literature, we show the variation in error with the waveguide tilt angle (see Fig. 3). We find that with only 900 computation points the overlap integral at all angles from 0 to 50 degrees is larger than 0.97. Thus, the accuracy is very high. At 50° the overlap integral value is almost 0.97, while Shibayama et al. obtained a value of ~0.97 with 2000 points in a multi-step, iterative procedure using a regular grid, or 1200 points with an adaptive grid. This clearly shows that the formulation we have presented here is more efficient in terms of fewer points required for computation and being non-iterative in nature, while yielding similar or better accuracy.

ACKNOWLEDGEMENT

This work was partially supported by a grant (No. 03(0976)/02/EMR-II) from the Council of Scientific and Industrial Research (CSIR), India.

REFERENCES


Fig. 2 Time required for computation of the $S$ matrix and propagation.

Fig. 3 Overlap integral with waveguide tilt angle.