Transmission Through Inhomogeneous Plane Layers*

J. H. RICHMOND†, SENIOR MEMBER, IRE

Summary—A plane wave is considered to be incident obliquely on a dielectric layer whose permittivity is a function of distance from the plane surface. The field distribution and the transmission coefficient are obtained from step-by-step numerical integration.

Since the calculations are simple and repetitive, the technique is suitable for both manual and automatic calculations. The method has been used successfully for lossy and lossless layers with perpendicular and parallel polarization. Some of the calculated field distributions are presented graphically. Only ten steps are required in the numerical integration for a typical half-wave inhomogeneous radome wall to obtain results accurate to within one per cent.

INTRODUCTION

THE PERMITTIVITY of most radome materials changes by a significant amount when the temperature is increased by hypersonic flight through the atmosphere. The outer surface of the radome becomes hotter than the inner, resulting in a continuous variation in permittivity even if the radome was designed as a homogeneous structure.

Moreover, new techniques of radome fabrication may make it feasible to construct continuously inhomogeneous radomes. This can be accomplished with variable loading or with variable density foams. Alternatively, a multilayer sandwich having many thin laminations can form an adequate approximation. These structures may have a greater bandwidth or a greater range of incidence angles than conventional radomes.

Exact solutions in closed form are available only for a few special cases including the linear and exponential inhomogeneities. An exact solution for more general cases is available in the form of a power series expansion, or an infinite series of integrals. However, the convergence is so slow that these techniques are practical only for layers that are thin in comparison with the wavelength.

Several techniques have been considered for solving the problem of plane-wave propagation through a continuously inhomogeneous layer. The layer may be approximated by a stack of homogeneous sheets; the matrix method or the standard sandwich formulas can then be applied. The problem can be formulated as an integral equation for which approximate solutions are available. Variational formulas have been developed.

The Riccati differential equation for the reflection coefficient or impedance can be used. The recent appearance of a book on the subject of inhomogeneous layers is indicative of the interest in this area.

In those cases where the permittivity gradient is small, the WKB asymptotic solution is most convenient for computing the field distribution or the transmission coefficient. Step-by-step numerical integration provides a practical solution even when the permittivity gradient is large.

The application of step-by-step numerical integration to inhomogeneous layers forms the subject of this paper. When used with the linear differential equations for the field intensity, this technique involves simple, repetitive calculations, and it yields the phase as well as the amplitude. When the solution for the field distribution has been completed, the transmission and reflection coefficients are readily determined.

Following a statement of the problem, techniques are described for starting the solution, completing the solution for the field distribution, and solving for the transmission coefficient.

I. Statement of the Problem

Consider a harmonic plane wave incident on a plane dielectric slab as in Fig. 1. The wave is said to have perpendicular polarization if the electric field intensity vector is perpendicular to the plane of incidence as in Fig. 1(a). It has parallel polarization if the electric field intensity vector is parallel with the plane of incidence as in Fig. 1(b). The time dependence e⁻ⁿᵗ is understood. All media are assumed to be linear and isotropic; the

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‡ Antenna Laboratory, Dept. of Electrical Engineering, The Ohio State University, Columbus, Ohio.
permittivity \(\varepsilon(z)\) is considered to be a function only of the distance \(z\) from the slab surface, and all the media have the same permeability \(\mu_0\).

The appropriate differential equation for perpendicular polarization, derived in Appendix I, is

\[
\frac{\partial^2 E}{\partial z^2} = -k^2(\varepsilon_r - \sin^2 \theta)E
\]

(perpendicular polarization) (1)

where \(E\) represents the \(x\) component of electric field intensity, \(k = \omega \sqrt{\mu_0 \varepsilon_0}\), \(\varepsilon_r\) is the complex relative permittivity, and \(\theta\) is the angle of incidence. The differential equations for parallel polarization are (see Appendix I):

\[
\frac{\partial H}{\partial z} = jw\varepsilon_0 k E
\]

(parallel polarization) (2)

\[
\frac{\partial E}{\partial z} = jk\sqrt{\mu_0 / \varepsilon_0}\frac{\varepsilon_r - \sin^2 \theta}{\varepsilon_r} H
\]

(parallel polarization) (3)

where \(H\) represents the \(x\) component of magnetic field intensity and \(E\) is the \(y\) component of electric field intensity. The field intensities involved in these equations are the components which are tangential to the slab surfaces. The boundary conditions require that they be continuous functions of \(z\), even at points where the permittivity is discontinuous.

Given the necessary data on the permittivity, the frequency, and the angle of incidence, it is required to solve for the field intensity distribution and the transmission coefficient to within some specified accuracy.

For perpendicular polarization,

\[
E_0 = 1
\]

\[
E_1 = 1 - \frac{k^2h^2}{2}(\varepsilon_r - \sin^2 \theta) - \frac{1}{6} k^2h^2\varepsilon_r
\]

\[+ j \left[ kh \cos \theta + \frac{k^2h^2}{2} \varepsilon_r \tan \delta_+ - \frac{k^2h^2}{6} (\varepsilon_r - \sin^2 \theta) \cos \theta \right] \] (5)

where \(\varepsilon_r\) is the real relative permittivity just inside the slab (at \(z=0\)), \(\tan \delta_+\) is the loss tangent just inside the slab, and \(\varepsilon'\) is the derivative of the permittivity with respect to \(z\). Eq. (5) is derived in Appendix II.

For parallel polarization the power series expansion is given by

\[
H_0 = 1
\]

\[
P_0 = \cos \theta
\]

\[
H_1 = 1 + kh\varepsilon_r \cos \theta \tan \delta_+ - \frac{k^2h^2}{2} (\varepsilon_r - \sin^2 \theta)
\]

\[+ jkh\varepsilon_r \left[ \cos \theta + \frac{kh}{2} \tan \delta_+ - \frac{1}{6} k^2h^2(\varepsilon_r - \sin^2 \theta) \cos \theta \right] \] (7)

\[
F_1 = \cos \theta + \frac{\sin^2 \theta}{\varepsilon_r} \tan \delta_+ - \frac{k^2h^2}{2} (\varepsilon_r - \sin^2 \theta) \cos \theta
\]

\[+ jkh \left[ \frac{\sin^2 \theta}{\varepsilon_r} + \frac{kh}{2} \cos \theta \tan \delta_+ - \frac{1}{6} k^2h^2 (\varepsilon_r - \sin^2 \theta)^2 + \frac{kh\varepsilon_r'}{2\varepsilon_r} \sin^2 \theta \right] \] (9)

where

\[
F_n = \sqrt{\varepsilon_0 / \mu_0} E_n.
\]

III. Completing the Solution for the Field Distribution

After the field has been determined at \(z=0\) and \(z=h\), step-by-step numerical integration can be used to calculate the fields \(E_n\) and \(H_n\) at the successive points \(z=nh\). If the second derivative in (1) is approximated by the quantity \((E_{n+1} - 2E_n + E_{n-1})/h^2\), the following second-order difference equation is obtained:

\[
E_{n+1} = \frac{[2 - k^2h^2(\varepsilon_r - \sin^2 \theta)]E_n - E_{n-1}}{} \]

(perpendicular polarization) (11)

where \(\varepsilon_n\) is the complex relative permittivity at \(z=nh\), and \(E_{n+1}\) and \(E_{n-1}\) are the field intensities at \(z=(n+1)h\) and \(z=(n-1)h\). It is convenient to resolve the complex
electric field intensity into real and imaginary components, $R_n$ and $I_n$ to obtain two real difference equations:

$$R_{n+1} = [2 - k^2h^2(\epsilon_n - \sin^2 \theta)]R_n - k^2h^2\epsilon_n I_n \tan \delta_n - R_{n-1},$$

$$I_{n+1} = [2 - k^2h^2(\epsilon_n - \sin^2 \theta)]I_n + k^2h^2\epsilon_n R_n \tan \delta_n - I_{n-1},$$

(12) (13)

where $\epsilon_n$ is the real relative permittivity and $\tan \delta_n$ is the loss tangent at $z = nh$. The fields $E_0$ and $E_1$ obtained from (4) and (5) may be inserted in (12) and (13) to calculate the field $E_2$ at the point $z = 2h$. Next, $E_1$ and $E_2$ are used in (12) and (13) to determine $E_3$. This process is continued until the slab has been traversed.

The difference equations for parallel polarization are

$$H_{n+1} = [2 - k^2h^2(\epsilon_n - \sin^2 \theta)]H_n + \frac{jkh^2\epsilon_n}{\epsilon_n^2}F_n - H_{n-1}$$

$$F_{n+1} = [2 - k^2h^2(\epsilon_n - \sin^2 \theta)]F_n + \frac{\frac{jkh^2\epsilon_n}{\epsilon_n^2} \sin^2 \theta}{H_n - F_{n-1}}.$$  

(14) (15)

Eqs. (14) and (15) can be used to obtain a system of four real equations. If the medium is lossless, the real equations are given by

$$X_{n+1} = K_n X_n - k^2h^2\epsilon_n V_n - X_{n-1}$$

$$Y_{n+1} = K_n Y_n + k^2h^2\epsilon_n U_n - Y_{n-1}$$

$$U_{n+1} = K_n U_n - k^2h^2(\epsilon_n/\epsilon_n^2) \sin^2 \theta Y_n - U_{n-1}$$

$$V_{n+1} = K_n V_n + k^2h^2(\epsilon_n/\epsilon_n^2) \sin^2 \theta X_n - V_{n-1}$$

(16) (17) (18) (19)

where

$$H = X + jY$$

$$F = U + jV = \sqrt{\epsilon_n/\mu_0} E$$

$$K_n = 2 - k^2h^2(\epsilon_n - \sin^2 \theta).$$

(20) (21) (22)

The starting values $H_0, H_1, F_0$ and $F_1$ obtained from (6) through (9) may be inserted in these equations to calculate $H_2$ and $F_2$.

A pair of simpler but less accurate difference equations can be obtained for parallel polarization by using the symmetrical differences $(H_{n+1} - H_{n-1})/2h$ and $(E_{n+1} - E_{n-1})/2h$ for the derivatives in (2) and (3).

IV. CHOOSING THE STEP SIZE

The power-series expansion for the field about the point $z = nh$ may be used to obtain the following:

$$E_{n+1} = E_n + E_n h + E_n'' h^2/2 + E_n''' h^3/6$$

$$+ E_n'''' h^4/24 + \cdots$$

$$E_{n-1} = E_n - E_n h + E_n'' h^2/2 + E_n''' h^3/6$$

$$+ E_n'''' h^4/24 + \cdots$$

(23) (24)

Addition of these two equations yields

$$E_{n+1} = 2E_n + h^2 E_n'' + h^4 E_n'''/12 - E_{n-1},$$

(25)

For perpendicular polarization, (1) may be used to reduce (25) to

$$E_{n+1} = [2 - k^2h^2(\epsilon_n - \sin^2 \theta)]E_n - E_{n-1} + h^4 E_n''''/12.$$  

(26)

This represents the difference equation for perpendicular polarization. Comparison with (11) shows that accurate results can be expected from step-by-step numerical integration using (11), (12) and (13) only if the neglected term $h^4 E_n''''/12$ is small. By differentiating both sides of (1) it can be shown that

$$h^4 E_n''''/12 = (h^4k^4/12)(\epsilon_n - \sin^2 \theta) E_n$$

$$- (h^4k^2/6) (\epsilon_n - \sin^2 \theta) E_n.$$  

(27)

The accuracy obtained with step-by-step numerical integration is limited by round-off error and neglect of the term $h^4 E_n''''/12$. The round-off error is determined by the number of significant figures carried, while the step size governs the error incurred in neglecting the term $h^4 E_n''''/12$. For economy of calculation time and effort, the two kinds of error should have the same order of magnitude.

If the purpose of the calculation is for comparison with experimental measurements, there is no point in making the round-off errors much smaller than the experimental errors. In particular, there usually are errors or uncertainties in the permittivity and the angle of incidence arising from manufacturing tolerances and measurement errors. An uncertainty $\Delta$ in the quantity $\epsilon_n - \sin^2 \theta$ induces an uncertainty of amount $k^2h^2 E_n \Delta$ in the calculation of $E_{n+1}$ by (26).

When the desired accuracy for the solution for $E(z)$ or $H(z)$ is specified, this makes possible a reasonable choice for the step size and the number of significant figures to be carried. If, for example, an over-all error $\xi$ is allowed, it might be assumed that an error of amount $K$ in each of $N$ steps is not likely to accumulate as over-all error of more than $NK$. Thus, let $K = \xi/N$. The calculation error in each step may be held to the allowed value $K$ by choosing the step size to make

$$h^4 E_n''''/12 = K,$$  

or more conveniently by making each of the three terms on the right-hand side of (27) equal to or less than $K$. Since $\|E\| = 1$ and $\|E'\| = k \sqrt{\epsilon_n - \sin^2 \theta}$, this approach leads to the following equations for the step size:

$$kh = 1.8K^{1/4}/\sqrt{\epsilon_n - \sin^2 \theta}$$

$$kh = 1.5[kK/(|\epsilon_n'| \sqrt{\epsilon_n - \sin^2 \theta})]^{1/4}.$$  

(28) (29)

For typical values of the allowable error $K$ in each step, these equations reduce to

$$0.32/\sqrt{\epsilon_n - \sin^2 \theta} \quad \text{for} \quad K = 0.001$$

$$0.18/\sqrt{\epsilon_n - \sin^2 \theta} \quad \text{for} \quad K = 0.0001$$

$$0.10/\sqrt{\epsilon_n - \sin^2 \theta} \quad \text{for} \quad K = 0.00001$$

(30)
If the above equations require a number of steps \( N = d/h \) considerably different from that assumed to specify the stepwise error \( K = \epsilon r / N \), the step size should be calculated again based on a revised estimate of \( K \).

If a constant step size is to be used throughout a layer, the step size should be determined by (28) and (29) on the basis of the maximum values of \( \epsilon_r \) and \( |\epsilon_r'| \) in the layer.

In practice, after the allowable error \( K \) is determined, the step size is chosen by means of (28) or (29), whichever requires the smaller steps. The step size is determined by (29) if \(|\epsilon_r'| > (k^2/2)(\epsilon_r - \sin^2 \theta)^{1/2}\), and by (28) otherwise.

A similar analysis for parallel polarization shows that (28) through (31) are suitable for choosing the step size for this case also.

In general, the step size used in starting the solution need not be the same as that used in continuing the solution. However, if (5), (8) and (9) are used, the step size given by (28) and (29) will be satisfactory for starting and continuing. The number of terms retained in (5), (8) and (9) was determined on this basis.

In radome design the thickness \( d \) is usually determined by \( kd \sqrt{\epsilon_r - \sin^2 \theta} = \pi \). The resulting "half-wave layer" has perfect transmission if the medium is lossless and homogeneous. This same design formula is a practical approximation for obtaining maximum transmission through an inhomogeneous layer if its average value of \( \sqrt{\epsilon_r - \sin^2 \theta} \) is inserted in the formula.\(^{11}\) If this design relation is used in (30) for \( K = 0.001 \), it is found that \( d/h = 10 \). That is, ten steps are required for step-by-step numerical integration through a "half-wave layer." The errors of 0.001 in each step might accumulate to an error of 0.01 in the final solution. Since the starting field is adjusted to a level of 1.0, the ten-step solution will provide an accuracy of approximately 1 per cent. If the permittivity gradient is large, however, (29) may require a larger number of steps for the same accuracy.

More refined expressions for the step size can be obtained by using the WKB approximate solutions\(^{11}\) for \( |E| \) and \(|E''|\) instead of the expressions for homogeneous media. These refined equations for the step size are found to agree closely with (28) and (29).

V. SOLVING FOR THE TRANSMISSION COEFFICIENT

When the solution for the field distribution is completed, it is quite simple to determine the transmission coefficient. If the slab thickness is denoted by \( d \), then \( z = d \) at the surface nearest the source. Let \((E_x, H_y)\) represent the tangential field components at the point (0, 0, 0), let \((E_i, H_i)\) be the tangential components of the incident plane wave at this same point, and let \((E_o, H_o)\) be the tangential field components at the exit point (0, 0, 0). Then the "normal transmission coefficient" is defined by

\[
T = E_o / E_i. \tag{32}
\]

For perpendicular polarization, \( E_o \) is taken to be unity. If \( R \) is the reflection coefficient,\(^{13}\) the following relations apply for perpendicular polarization:

\[
E_d = (1 + R)E_i, \tag{33}
\]
\[
H_d = (1 - R)H_i. \tag{34}
\]

Furthermore,

\[
H_i = \sqrt{\epsilon_0 / \mu_0} E_i \cos \theta \tag{35}
\]
and

\[
H = (-j / \omega \mu_0) \partial E / \partial z. \tag{36}
\]

These equations yield the following result for the transmission coefficient:

\[
T = \frac{2}{E_d - jE_d' / (k \cos \theta)} \quad \text{(perpendicular polarization),} \tag{37}
\]

\( E_d' \) can be approximated by \( [E(d) - E(d - h)] / h \), or more accurately by the symmetrical difference \( [E(d + h) - E(d - h)] / 2h \).

For parallel polarization, the transmission coefficient is given by

\[
T = \frac{2}{H_d + F_t / \cos \theta} \quad \text{(parallel polarization).} \tag{38}
\]

VI. EXAMPLES

The expressions presented herein have been employed to calculate the field distributions in homogeneous and inhomogeneous layers. In the examples selected, the exact solutions are also available as a check on the accuracy. Some of the results are shown in Fig. 2 in the form of the polar loci of the electric field intensity. For an error \( K = 0.0001 \) per step, (30) calls for a step size corresponding to \( kh = 0.1 \) for the examples illustrated. This step size was used in all cases, and the step-by-step numerical integration was carried out through 36 steps, although only 30 steps are shown in Fig. 2. Comparison with the exact solutions shows that the accumulated error \( \mathcal{E} \) in each case is less than \( 3NK \) which is about 0.01. (\( N \) represents the number of steps.) Thus, the accuracy of these solutions is within one per cent.

Eqs. (16) through (22) have been used for analyzing the field distribution for parallel polarization for an angle of incidence of 60 degrees. The relative permittivity varied exponentially from 8 to 10 in a layer having a thickness of \( d = 0.8/k \). A step size corresponding to

\(^{11}\) The symbol \( R \) used here for the reflection coefficient should not be confused with the same symbol used elsewhere in this paper for the real part of the field intensity.
The solution by step-by-step numerical integration with the WKB solution shows excellent agreement. Although no exact solution is available for this problem, comparison with the WKB solution shows excellent agreement.11

Table I shows the real and imaginary parts of the electric field distribution in a homogeneous slab as calculated with the equations developed herein. The exact solution is also tabulated for comparison. Five significant figures were carried in the calculations, and the results were later rounded off to four figures for Table I. The solution by step-by-step numerical integration took about 15 minutes on a desk calculator. The data in Table I represent part of the solution for curve (a) in Fig. 2.

![Electric Field Distribution in a Homogeneous Slab](image)

**FIG. 2.—Polar loci of the electric field intensity in inhomogeneous plane slabs for normal incidence.**

**TABLE I**

<table>
<thead>
<tr>
<th>Electric Field Distribution in a Homogeneous Slab</th>
</tr>
</thead>
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<tr>
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<table>
<thead>
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<th>Step-by-step Solution</th>
<th>Equation Number</th>
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<td>1.0000</td>
<td>4</td>
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<tr>
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**CONCLUSIONS**

Step-by-step numerical integration provides a convenient solution for the field distribution in an inhomogeneous plane slab. A power-series expansion is used to calculate the field at a point a short distance into the slab. This provides a starting point for calculating the field at equally spaced points through the slab. When this is completed, the transmission coefficient can readily be determined.

The necessary equations are included for both perpendicular and parallel polarization. These have been applied to layers having linear and exponential variations in permittivity, and the results are shown graphically.

Equations are developed for the step size required to obtain a specified accuracy. Short steps are required if the permittivity or its gradient is large. If the fields are to be calculated to an accuracy within one per cent, only ten steps are needed for a “half-wave slab.”

In the solution for a sandwich having both continuous and discontinuous variations in permittivity, the step-by-step integration must be interrupted at each interface to calculate the next starting point. Power-series expansions similar to those in Section II can be derived for this purpose.

The techniques described are also applicable in solving the following problems:

1) A dielectric-filled waveguide in which the permittivity is a function of distance along the waveguide axis.

2) A cylindrical wave incident on a circular dielectric cylinder whose permittivity is a function of distance from the cylinder axis.

3) A spherical wave incident on a dielectric sphere whose permittivity is a function of distance from the center.

**APPENDIX I**

**THE DIFFERENTIAL EQUATIONS**

Since

\[ \nabla \cdot D = \nabla \cdot (\varepsilon E) = E \cdot \nabla \varepsilon + \varepsilon \nabla \cdot E = 0, \tag{39} \]

the wave equation

\[ \nabla \times \nabla \times E = \nabla (\nabla \cdot E) - \nabla^2 E = \omega^2 \mu \varepsilon E \tag{40} \]

can be written in the form

\[ \nabla \left( \frac{E \cdot \nabla \varepsilon}{\varepsilon} \right) + \nabla^2 E = - \omega^2 \mu \varepsilon E. \tag{41} \]

If \( \varepsilon \) is a function of \( z \) only,

\[ E \cdot \nabla \varepsilon = E_0 \varepsilon'. \tag{42} \]

For perpendicular polarization \( E_z = 0 \) and (41) reduces to

\[ \nabla^2 E_z = - \omega^2 \mu \varepsilon E_z = - k^2 \varepsilon E_z. \tag{43} \]

If the fields vary as \( e^{ikx} \sin \theta \) and if \( \partial / \partial x = 0 \), (43) reduces to (1).

For parallel polarization \( H \) has only an \( x \) component denoted by \( H \). Maxwell’s equations yield (2) directly and

\[ \omega E_z = - \partial H / \partial y = - k H \sin \theta \tag{44} \]

\[ -j \omega H = \partial E_y / \partial y - \partial E_x / \partial z = jk E_z \sin \theta - \partial E_y / \partial z. \tag{45} \]

These two equations can be solved for \( \partial E_y / \partial z \) to obtain (3).
APPENDIX II

The Power-series Expansion for Perpendicular Polarization

The power-series expansion for $E$ in the region $0 \leq z$ is given by

$$E = E_0 + E'_0 z + E_0'' s^2/2 + E_0''' s^3/6 + E_0'''' s^4/24 + \cdots \quad (46)$$

where $E'_0$ represents $\partial E/\partial z$ at $y=0$ and $z=0+$. This expansion is valid in the region $0 < z < h$ if the permittivity and its derivatives are continuous there.

From Maxwell's equations

$$\partial E/\partial z = E' = -j \omega \mu H_y. \quad (47)$$

Since $\mu$ and $H_y$ are continuous functions of $z$, it is evident from (47) that $E'$ is continuous. For $z < 0$,

$$E = E_0 e^{j \theta z} \cos \theta \phi e^{j \phi} \sin \theta. \quad (48)$$

Hence

$$E_0' = j E_0 k \cos \theta. \quad (49)$$

From (1),

$$E_0'' = -k^2(\epsilon_+ - \sin^2 \theta) E_0. \quad (50)$$

Differentiation of (1) yields

$$E_0''' = -k^3[\epsilon_+ - \sin^2 \theta) \cos \theta + \epsilon_+'] E_0. \quad (51)$$

$$E_0'''' = -k^4(\epsilon_+ - \sin^2 \theta)^2 + 2j k \epsilon_+ \cos \theta + \epsilon_+' E_0. \quad (52)$$

Eq. (5) is obtained by letting $z = h$ in (46), using (4), (49)-(52), and neglecting the terms that will be negligible in view of the small step size. [See (28) and (29).]

Charts for Computing the Refractive Indexes of a Magneto-Ionic Medium

G. A. DESCHAMPS†, FELLOW, IRE, AND W. L. WEEKS‡, MEMBER, IRE

Summary—The complex refractive index of a magneto-ionic medium for various directions of propagation is evaluated graphically. Most of the constructions make use of a Smith Chart on which the scales have been suitably relabeled. When the directions of propagation are longitudinal or transverse with respect to the applied magnetic field, the constructions take the losses into account. The energy transferred to the medium is characterized by conductivity coefficients that are also evaluated graphically. An overlay converts the refractive index into the reflection coefficient at a plane boundary of the medium. The indexes for propagation at an arbitrary angle with respect to the magnetic field are evaluated similarly, provided the collision frequency is very low.

1. Introduction

There is a need in some applications for a rapid evaluation of the characteristics of a plane wave propagating through a magneto-ionic medium. The attenuation, the wavelength, and the reflection coefficient at a boundary of the medium may be desired as functions of the electron density, the collision frequency, the applied magnetic-field vector, and the direction of propagation.

These characteristics can be derived from the Appleton-Hartree equation but, although this equation is only of the second degree and involves nothing but simple trigonometry, the number of basic variables and the necessity for computing with complex numbers make it rather unwieldy.

A digital computer could of course be used but when only moderate accuracy is needed, due to the nature of the data, graphical methods can be superior. Once a few basic charts have been prepared they give results quickly and economically. The effects of changes in the basic variables are immediately seen and a whole field of results can be displayed at once and grasped at a glance.

The advantages of graphical methods have been well recognized in other fields. The solution of transmission line problems by means of the Smith Chart [5] is an outstanding example: the conversion of reflection coefficient into impedance is performed by inspection; a complicated matrix multiplication, corresponding to a change of reference plane, is replaced by a rotation about the center of the chart, and numerous problems of matching and filtering can be solved by simple constructions.

In the field of magneto-ionic theory important work in this direction has been done by Bailey [1] who has given several graphical constructions for the polariza-

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† University of Illinois, Urbana, Ill.
‡ Collins Radio Company, Dallas, Tex. Formerly with the University of Illinois, Urbana, Ill.