Inverse Scattering Method for One-Dimensional Inhomogeneous Layered Media

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Abstract—An inverse scattering method to reconstruct simultaneously the permittivity profile and the conductivity profile of one-dimensional inhomogeneous medium which makes use of the transverse electric (TE) wave and/or transverse magnetic (TM) wave, is proposed. The medium is illuminated by the TE and/or TM plane wave at oblique incidence, and the data are taken as the reflection coefficients for a set of discrete frequencies and/or a finite number of incident angles. Furthermore, the reflection coefficient data contain the Gaussian noise. The nonlinear methods may be generally classified into two approaches. The first approach is inverse mapping. Generalization or modification of Gel'fand–Levitan type equation is representative of this approach. The best fitting method or the iterative procedure is the second approach.

The inverse scattering problem for reconstructing the permittivity profile of an inhomogeneous medium results in the inverse potential problem of Schroedinger equation through a Liouville transformation. This problem is uniquely solvable. However, the solution technique of the Gel'fand–Levitan equation and the derivation from the potential function to the permittivity profile involves considerable mathematical complexity, so the exact solutions have been obtained only for the problems under the very restricted conditions [7]–[9]. Balanis has solved numerically the one-dimensional plasma inverse scattering problem [10], but this problem is essentially identical to the inverse potential problem of the Schroedinger equation. Weston and Krueger have formulated the Gel'fand–Levitan type equation for lossy layered medium [11]–[14]. The permittivity profile and the conductivity profile can be uniquely determined by solving the simultaneous integral equation with four unknown kernels which are derived from all components of scattering matrix in the time domain. Furthermore, Krueger et al. [15]–[17] have developed this approach and have given a new algorithm for determining the permittivity and the conductivity of lossy slab from the measurement data in time domain. They have also discussed the noisy data. On the other hand, Tabbara has proposed a different method for a lossless dielectric slab in the frequency domain [18]. The permittivity profile is given by the closed form. This method has been derived from the improvement of the Born approximation.

The second inverse scattering approach is usually formulated by the source-type integral equation which relates the constitutive parameter of a medium to the scattering field. The permittivity profile of a lossless dielectric slab terminated by a perfectly conducting boundary has been reconstructed by the successive minimization of a squared error of reflection coefficient for a set of discrete frequencies or a finite number of angles of incidence of a monochromatic transverse electric (TE) plane wave [19]. The permittivity and conductivity profiles of the lossy slab have been reconstructed by using the quasi-Newton method from the knowledge of the multifrequency reflected power measured at the front of the slab for normal incident wave [20]. The convergence and the stability of this iterative computation are not good because the phase information of the reflection coefficient is lacking.

An alternative approach is formulated by the subsequently discretization of the source-type integral equation by using the trapezoidal quadrature rule in time or frequency domain. Various methods have been studied: 1) step-by-step determination of the unknown profile by tracing the wave front of the field [21], [22]; 2) the iteration method which consists of successively computation of a direct scattering problem and an approximate inverse scattering problem [23]; 3) the iterative optimization approach based on the regularization method [24], [25]. These inverse methods are restricted to the case of no-loss medium, or the case where one of the constitutive parameters of medium is a priori known.

A feature of the electromagnetic inverse scattering problem is the fact that the polarization of incident wave can be utilized. In this paper, an inverse scattering method to reconstruct the constitutive parameters of one dimensional inhomogeneous medium which makes use of the TE wave, or/and transverse magnetic (TM) wave and the variation of incident angle, is proposed [26]. The permittivity profile and the conductivity profile are reconstructed simultaneously from the reflection

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coefficient data of TE wave and TM wave for a set of discrete frequencies and/or a finite number of incident angles. The unknown profile is reconstructed by solving the nonlinear integral equation in terms of the Newton iterative procedure. The inverse operator used in the iteration process is determined by the regularization method. The successively increasing multiple layer method is proposed as a very effective iteration method. The reflection coefficient data used for the computer simulation contain the Gaussian noise for the sake of the practical applications. The relationships between the errors of the reconstructed profile and the reflection coefficient are also discussed.

II. THE BASIC NONLINEAR INTEGRAL EQUATIONS

The exact nonlinear integral equations for TE wave incidence and TM wave incidence are derived in this section. Each integral equation relates the constitutive parameters of an inhomogeneous layered medium to the reflection coefficient.

The geometry of the problem is illustrated in Fig. 1. The permittivity $\varepsilon^*(x)$ and the conductivity $\sigma^*(x)$ are independent of the frequency and are the function of depth $x$ only. $\varepsilon_0$ and $\sigma_0$ are assumed to be known constants.

A. TE Wave Incidence

Suppose that the TE plane wave having $y$-component electric field $E_y$ is incident on the medium making angle $\theta$ with the $x$-axis. Then

$$E_y(x, z) = \phi^*(x) e^{z j \sin \theta}$$

satisfies the differential equation

$$\frac{d^2 \phi^*}{dx^2} + k^2 \{q^*(k, x) - \sin^2 \theta \} \phi^* = 0$$

(1)

where $q^*(k, x) = \varepsilon^*(x) - j(\eta_0/k)\sigma^*(x)$ is a complex permittivity, $k$ is a wavenumber of free space, and $\eta_0 = (\mu_0/\varepsilon_0)^{1/2}$ is a characteristic impedance of free space. The electric field in the homogeneous regions $x < 0$ or $x > a$ is given by

$$\phi^*(x) = \begin{cases} e^{z j \cos \theta + r_{TE}(q^*) e^{j k \cos \theta}}, & x \leq 0, \\ t_{TE}(q^*) e^{-z j \sqrt{\varepsilon_e - \sin^2 \theta} (x-a)}, & x \geq a, \end{cases}$$

(2)

where $r_{TE}(q^*)$ and $t_{TE}(q^*)$ are the reflection coefficient and the transmission coefficient, respectively, and $\varepsilon_e = \varepsilon_0 - j(\eta_0/k)\sigma_0$.

From (1) we obtain

$$\frac{d}{dx} \left( \phi^* \frac{d \phi^*}{dx} - \phi \frac{d \phi^*}{dx} \right) = k^2 (q^* - q) \phi^*$$

(3)

where $\phi$ is the electric field for another complex permittivity $q(k, x)$. Integrating (3) over the inhomogeneous region $0 < x < a$, and using (2) and the boundary conditions at $x = 0, a$, the reflection coefficient $r_{TE}(q^*)$ is obtained as follows:

$$r_{TE}(q^*) = r_{TE}(q) + \frac{k}{j 2 \cos \theta} \int_0^a \left\{ q^*(k, x) - q(k, x) \right\} \cdot \phi^*(x) \phi(x) \, dx.$$  (4)

B. TM Wave Incidence

The integral equation for TM wave incidence can be obtained by the similar manner to the TE wave incidence. The $y$-component magnetic field $H_y(x, z) = \psi^*(x) e^{z j \sin \theta}$ satisfies

$$\frac{d}{dx} \left( \frac{1}{q^*} \frac{d \psi^*}{dx} \right) + k^2 \left( 1 - \sin^2 \theta \right) \frac{\psi^*}{q^*(k, x)} = 0.$$  (5)

The magnetic field in the homogeneous region $x < 0$ or $x > a$ is given by

$$\psi^*(x) = \begin{cases} \frac{1}{\eta_0} \{ e^{-z j \cos \theta + r_{TM}(q^*) e^{j k \cos \theta}} \}, & x \leq 0, \\ \sqrt{\varepsilon_e} t_{TM}(q^*) e^{-z j \sqrt{\varepsilon_e - \sin^2 \theta} (x-a)}, & x \geq a. \end{cases}$$  (6)

From (5), we obtain the following relation:

$$\frac{d}{dx} \left( \psi^* \left( \frac{1}{q} \frac{d \psi}{dx} \right) - \psi \left( \frac{1}{q^*} \frac{d \psi^*}{dx} \right) \right) = \left( \frac{1}{q} - \frac{1}{q^*} \right) \left( \frac{d \psi}{dx} \frac{d \psi^*}{dx} + k^2 \sin^2 \theta \psi \psi^* \right).$$  (7)

Integrating (7) over the inhomogeneous region $0 < x < a$, and using (6) and the boundary conditions at $x = 0, a$, the integral equation for TM wave incidence is obtained

$$r_{TM}(q^*) = r_{TM}(q) + \frac{2 \eta_0}{j 2 k \cos \theta} \int_0^a \left\{ \frac{1}{q(k, x)} - \frac{1}{q^*(k, x)} \right\} \left\{ \frac{d \psi}{dx} \frac{d \psi^*}{dx} + k^2 \sin^2 \theta \psi \psi^* \right\} \, dx.$$  (8)

Let $q(k, x)$ be any given profile, then $\phi(q, x), \psi(q, x), r_{TE}(q)$ and $r_{TM}(q)$ can be calculated from the direct scattering problem. Suppose $r_{TE}(q^*)$ and $r_{TM}(q^*)$ are the observed reflection coefficient data, then (4) and (8) are the nonlinear integral equations because that $\phi^*(q^*, x)$ and $\psi^*(q^*, x)$ are also the function of unknown profile $q^*(k, x)$. 

Fig. 1. Geometry of the problem. $\varepsilon_0$ and $\sigma_0$ are known constants.
The permittivity profile $\varepsilon(x)$ and the conductivity profile $\sigma(x)$ are determined in principle by inverting the integral equation (4) or (8). However, it is not possible in general to obtain the unique solution of the nonlinear integral equation. Furthermore, since the measured reflection coefficients in practical applications are always given for a finite number of frequencies and/or incident angles, there can be an infinite number of solutions [27], [28]. Therefore, the most reasonable solution must be determined numerically in the sense of the best fitting. In this paper, the Newton iterative method is adopted, and the regularization method is used to determine the inverse operator in the iterative procedure under the consideration of above situation.

Let us assume that the unknown profile $q^*$ is given by a small perturbation from the known profile $q$, i.e.,

$$ q^*(k, x) = q(k, x) + \delta q(k, x). \tag{9} $$

The corresponding perturbations of the electric field $\phi$ and the magnetic field $\psi$ in the inhomogeneous region are expressed by $\delta \phi$ and $\delta \psi$, respectively. The substitution of (9) into (5) or (8) and the neglect of the term $\|\delta q\|^2$ lead to the following approximate linear integral equation.

$$ r(q^*) = r(q) + L \delta q, \tag{10} $$

where

$$ L \delta q = \int_0^a F(q, k, x) \delta q \, dx \tag{11} $$

and

$$ F(q, k, x) = \begin{cases} \dfrac{k}{j2 \cos \theta} \phi^2(q, x); & \text{TE} \\ \dfrac{\eta_0^2}{j2k \cos \theta} \dfrac{1}{q^2(k, x)} \left( \dfrac{d \psi}{dx} \right)^2 + k^2 \sin^2 \theta \psi^2; & \text{TM}. \tag{12} \end{cases} $$

The Newton iterative method is presented as follows:

$$ q_l+1 = q_l + L^{-1} \{ r^* - r_l(q_l) \}, \quad l = 1, 2, \cdots, \tag{13} $$

where $l$ is the iteration number, $r^*$ is the known observed reflection coefficient, and $r_l(q_l)$ is the computed reflection coefficient from the known profile $q_l$. $q_l$ is expected to converge to the real profile $q^*$ for large $l$.

The inverse operator $L^{-1}$ is determined by the regularization method [29], [30] in this paper, because of the fact that the reflection coefficient data always contain the error and are given only for a set of discrete frequencies and/or a finite number of incident angles. The fundamental equation is formulated as follows:

$$ T(q) = S(q) + \|\delta q\|^2 \rightarrow \min \tag{14} $$

where

$$ S(q) = \sum_{i=1}^n W_i |r_i^* - r_i(q)|^2, \tag{15} $$

$$ \|\delta q\|_2^2 = \int_0^a \left\{ W_{e}(x) \delta^2(x) + W_{s}(x) \delta^2(x) \right\} \, dx. \tag{16} $$

In (15) $r_i^*$ ($i = 1, 2, \cdots, n$) are the observed reflection coefficient data for a set of discrete frequencies and/or a finite number of incident angles of TE wave and/or TM wave. $W_i, W_{e}(x)$, and $W_{s}(x)$ are positive weighting functions. In the concrete numerical examples which will be shown in the following section, $W_i$ is chosen as $W_i = 1/(k_i a)^2$, but the remarkable difference in the convergence from alternative choice such as $W_i = 1$ cannot be recognized. $W_{e}(x)$ and $W_{s}(x)$ are chosen as $W_{e}(x) = W_{s}(x) = \text{const.}$ which is a Marquardt type constant [31].

The minimization of (14) can be performed analytically. The resultant expression is given as follows:

$$ W_{e}(x) \delta \varepsilon(x) + \int_0^a K_{11}(x, x') \delta \varepsilon(x') \, dx' + \int_0^a K_{12}(x, x') \delta \sigma(x') \, dx' = C_1(x) \tag{17} $$

$$ W_{s}(x) \delta \sigma(x) + \int_0^a K_{21}(x, x') \delta \varepsilon(x') \, dx' + \int_0^a K_{22}(x, x') \delta \sigma(x') \, dx' = C_2(x) $$

where

$$ K_{11}(x, x') = \text{Re} \left\{ \sum_{i=1}^n W_i F(q, k_i, x) F(q, k_i, x') \right\} $$

$$ K_{12}(x, x') = \text{Im} \left\{ \sum_{i=1}^n W_i F(q, k_i, x) F(q, k_i, x') \right\} $$

$$ K_{21}(x, x') = - K_{12}(x, x') $$

$$ K_{22}(x, x') = \frac{\eta_0^2}{a^2} \text{Re} \left\{ \sum_{i=1}^n \frac{W_i}{k_i} F(q, k_i, x) F(q, k_i, x') \right\} $$

$$ C_1(x) = \text{Re} \left\{ \sum_{i=1}^n W_i (r_i^* - r_i(q)) F(q, k_i, x) \right\} $$

$$ C_2(x) = \text{Im} \left\{ \sum_{i=1}^n \frac{W_i}{k_i} (r_i^* - r_i(q)) F(q, k_i, x) \right\} \, a. \tag{18} $$

The simultaneous integral equation (17) is a second kind Fredholm type integral equation with degenerated kernel. Consequently, $\delta \varepsilon(x)$ and $\delta \sigma(x)$ can be uniquely solved. The
resultant expressions are given by

\[
\delta e(x) = \frac{1}{a W_t(x)} \left[ C_1(x) - \sum_{i=1}^{n} W_i \{ \alpha e f(k_i, x) + \beta e g(k_i, x) \} \right]
\]

\[
\delta o(x) = \frac{a}{\eta_0 W_t(x)} \left[ C_2(x) - \frac{1}{a} \sum_{i=1}^{n} W_i \{ \alpha o g(k_i, x) \} - \beta o f(k_i, x) \right]
\]

(19)

\[
f(k_i, x) = \text{Re} \{ aF(q, k_i, x) \}, \quad g(k_i, x) = \text{Im} \{ aF(q, k_i, x) \}
\]

where the coefficients \( \alpha e \) and \( \beta e \) are given in Appendix.

If the medium consists of homogeneous multilayers and the depth of each layer is known, the above process is performed in vector space.

**IV. Numerical Examples for Multifrequency Inverse Scattering Method**

The examples of the numerically reconstructed permittivity and conductivity profiles from the artificial data are shown in this section. The data are artificial in the sense that the reflection coefficients for TE wave incidence and TM wave incidence are simulated on the computer for a given profile. Suppose that in the geometry shown in Fig. 1, the depth \( a \), the dielectric constant \( \epsilon_r \), and the conductivity \( \sigma \) of the homogeneous half-space \( x > a \) are *a priori* known.

It is inevitable in practice that the observed reflection coefficient data contain a certain amount of measurement error. In order to simulate such a situation, the data for reconstruction are given as follows:

\[
r_i^{\text{true}} = r_i + \delta(n_i^r, \eta_i^m), \quad (i = 1, 2, \cdots),
\]

(20)

where \( r_i \) is the exact reflection coefficient calculated by the direct scattering problem, and \( n_i^r \) and \( \eta_i^m \) are assumed as real and imaginary zero-mean Gaussian random variable of variance 1.

The solution of the iterative method depends on the initial guess as well as the measurement data. Hence, the initial guess must be selected carefully for the sake of good convergence and stable computation. In this paper, the initial guess is given by the following method. First, suppose that the medium is homogeneous, then the permittivity and the conductivity of the medium are reconstructed by the above procedure in vector space. The initial guess in this case is taken as the vacuum.

The reconstructed profile by using the above method from the reflection coefficients for \( ka = 0.1, 0.2, \cdots, 1.5 \) is shown in Fig. 2. It is found that the reconstructed profile for \( m = 1 \) converges to about an average value of real profile. This method can be used not only in the determination of the initial guess but also in the reconstruction of the profile for the cases that the depth of multilayer medium is unknown, or the medium is continuous. The dotted lines show the profiles reconstructed under the assumption that the medium consists of ten layers from the beginning of the iteration and the initial guess of the iteration is taken as the vacuum.

Fig. 3 illustrates the profiles reconstructed from the reflection coefficient data for TE wave. The data are 15 reflection coefficients for \( ka = 0.1, 0.2, \cdots, 1.5 \). The real profiles are also shown for comparison. It is found that the reconstruction can be performed very precisely for free error case (\( \delta = 0 \)). In this example, the initial guess of the Newton method was the homogeneous medium calculated by the above method, that is, the profile for \( m = 1 \). The direct scattering program used in the iteration is calculated by using the 200-division homogeneous multilayer approximation method.

Fig. 4 shows the case that the medium has four homogeneous layers, and the incident wave is TE. The reconstructed profiles for \( \delta = 10^{-3} \) are not good.

**V. Utilization of Both TE Wave and TM Wave**

As shown above, if the error is very small, the multifrequency inverse scattering method for TE wave incidence or TM wave incidence can reconstruct successfully the real profile. However, if the error variance increases to about \( 10^{-3} \) or more, then the inverse scattering procedure converges to the quite different profile from the real one. In order to improve this, a new inverse scattering method making use of both TE wave and TM wave with varied incident angle is proposed in this paper. The fundamental equation is

\[
T(q) = S_{\text{TE}}(q) + S_{\text{TM}}(q) + \| \delta q \|_2^2 \rightarrow \text{min}.
\]

(21)

Since the minimization procedure of (21) and the derivation of the reconstructed profile are similar to that in Section III, the procedures are not shown here. In order to confirm the effectiveness of this method, the reconstruction is performed for the simple homogeneous multilayer medium.

Fig. 5 shows the profile reconstructed from the reflection coefficient data of both TE wave and TM wave for a single incident angle \( \theta = 30^\circ \) which are shown in Fig. 6. By comparing with Fig. 4, it is found that the improvement can be attained by utilizing TE wave and TM wave simultaneously.

Fig. 7 illustrates the reconstructed profile in the case where the reflection coefficient data of TE wave and TM wave are known not only for \( ka = 0.1, \cdots, 1.5 \) but also for the varied incident angles \( \theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ \). It is found that a great improvement can be attained in comparison with Figs. 4 and 5.

The case that only the amplitude of the reflection coefficient
Fig. 2. Determination method of initial guess for Newton iterative method. \( m \) indicates the number of layer. The reflection coefficient data of TE wave incidence are known for \( k\alpha = 0.1, 0.2, \ldots, 1.5 \) and \( \theta = 60^\circ \). (a) \( m = 1, 2 \). (b) \( m = 10 \).
Fig. 3. Reconstructed profile of continuous medium. The reflection coefficient data of TE wave incidence are known for $ka = 0.1, 0.2, \cdots, 1.5$ and $\theta = 30^\circ$.

Fig. 4. Reconstructed profile of the medium having four homogeneous layers. The reflection coefficient data of TE wave incidence is given for $ka = 0.1, 0.2, \cdots, 1.5$ and $\theta = 30^\circ$. 
is known is shown in Fig. 8. Reconstruction is inferior to the case of Fig. 7. However, the reconstructed profiles are still good in comparison with the case of Fig. 4.

VI. ERROR ESTIMATION OF RECONSTRUCTED PROFILE

The inverse scattering method proposed in this paper is very effective even if the reflection coefficient contains some practical degrees of the measurement error. However, one cannot neglect the influence of the measurement error on the reconstructed profile. It is important therefore to clarify the degree to which the measurement error affects the reconstructed profile. In this section, the relationships of the errors between the reflection coefficient and the reconstructed profile are discussed.

Let \( q^* \) and \( R \) be real profile and the exact reflection coefficient, respectively. \( q \) is assumed to be profile reconstructed from \( r \) which contains the error. From (10)

\[
\delta r = R - r = L \delta q. \tag{22}
\]

Applying the Schwarz inequality, we can obtain the following expression:

\[
\| \delta \epsilon \|^2 = \frac{1}{\beta a^2} \left\{ \sum_{i=1}^{n} \frac{1}{k_i^2} \| f(k_i, x) \|^2 \sum_{i=1}^{n} | \rho_r(k_i) |^2 - \sum_{i=1}^{n} \frac{1}{k_i^2} \| g(k_i, x) \|^2 \sum_{i=1}^{n} | \rho_m(k_i) |^2 \right\}
\]

\[
\| \delta \sigma \|^2 = \frac{1}{a^2 \beta \eta_0^2} \left\{ - \sum_{i=1}^{n} \| g(k_i, x) \|^2 \sum_{i=1}^{n} | \rho_r(k_i) |^2 + \sum_{i=1}^{n} \| f(k_i, x) \|^2 \sum_{i=1}^{n} | \rho_m(k_i) |^2 \right\}
\]

\[
\beta = \frac{1}{a^2} \left\{ \sum_{i=1}^{n} \| f(k_i, x) \|^2 \sum_{i=1}^{n} \| f(k_i, x) \|^2 - \sum_{i=1}^{n} \| g(k_i, x) \|^2 \sum_{i=1}^{n} \frac{1}{k_i^2} \| g(k_i, x) \|^2 \right\}
\]

\[
\rho_r = \text{Re} \ (r^* - r), \quad \rho_m = \text{Im} \ (r^* - r). \tag{23}
\]

If the error conforms to the Gaussian distribution, we obtain

\[
\| \delta \epsilon \|^2 = \frac{n}{\beta a^2} \left\{ \sum_{i=1}^{n} \frac{1}{k_i^2} (\| f(k_i, x) \|^2 - \| g(k_i, x) \|^2) \right\} \delta^2
\]

\[
\| \delta \sigma \|^2 = \frac{n}{\beta a^2 \eta_0^2} \left\{ \sum_{i=1}^{n} (\| f(k_i, x) \|^2 - \| g(k_i, x) \|^2) \right\} \delta^2. \tag{24}
\]

The error variances of the reconstructed permittivity profile and the conductivity profile could be estimated in advance from the error variance of the given reflection coefficient by (24). The numerical example of the estimation of the error variance of the reconstructed profile is shown in Fig. 9 for the single layer under the following conditions:

TE wave incidence, \( \theta = 30^\circ \),

\[
ka = 0.1, 0.2,
\]

\( \epsilon^*(x) = 5, \sigma^*(x) = 10 \) (mS). \tag{25}

The dots indicate the actual error variance of the reconstructed profile. The solid line indicates the error variance estimated from (24). It is found that the (24) gives approximate upper bound of the error variance of the reconstructed profile.

VII. CONCLUSION

An inverse scattering method to reconstruct simultaneously the permittivity profile and the conductivity profile of one-
Fig. 6. Given reflection coefficient data. (a) TE wave incidence. (b) TM wave incidence.
Fig. 7. Reconstructed profile in the case that the reflection coefficient data of TE wave and TM wave are known not only for $ka = 0.1, 0.2, \cdots, 1.5$ but also for $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ$.

Fig. 8. Reconstructed profile in the case that only the amplitude of the reflection coefficient is known. Another conditions are same as the case of Fig. 7.
Fig. 9. Error variance of the reconstructed profile versus error variance of the given reflection coefficient data. The dots indicate the real error variance. The solid line indicates the estimated error variance. (a) Error variance of the permittivity. (b) Error variance of the conductivity.

dimensional inhomogeneous medium which makes use of the TE wave, TM wave or both with the varied incident angle, has been proposed. The data are taken as the reflection coefficients containing the Gaussian noise for a set of discrete frequencies and/or a finite number of incident angles of TE wave and TM wave.

The nonlinear integral equation relating the unknown constitutive parameter of the medium and the reflection coefficient has been solved by the Newton iterative method in vector space for homogeneous multilayer medium and in function space for the continuous medium. The inverse operator in the Newton iterative process has been determined by the regularization method. The permittivity and the conductivity have been determined by linking the Newton iteration process to the successively increasing multiple layer method which has been proposed in Section IV. In this iteration process, the initial guess has been taken as the vacuum.

An inverse problem contains very difficult problems in general. First, it can not be guaranteed that nonlinear problems such as the one treated in this paper has unique solution. Secondarily, for a practical case where the data set is finite and the function to be minimized is defined over an interval there could be an infinite number of solutions [27], [28]. Thirdly, the solution that is arrived at in the iteration process depends possibly upon the initial guess as well as the choice of measurements.

Notwithstanding that the above mentioned essential problems remain unanswered, the present paper has demonstrated that our method is practically very effective even if the reflection coefficient contains the practical degree of the measurement error, or even when the phase of the reflection coefficient is unknown.

It has been also indicated that the error variance of the reconstructed profile can be effectively estimated in advance from the error variance of the given reflection coefficient data.

APPENDIX

Equation (19) is obtained from the ordinary solution technique of the second kind Fredholm integral equation with the degenerated kernel. The coefficients \( \alpha_i \) and \( \beta_i \) are given by the following matrix equation:

\[
\begin{pmatrix}
I + A & B \\
C & I + D
\end{pmatrix}
\begin{pmatrix}
\hat{\alpha} \\
\hat{\beta}
\end{pmatrix}
= \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
\hat{\rho}_i \\
\hat{\rho}_m
\end{pmatrix}
\] (26)

where the caret indicates a column vector, \( I \) is a unit vector and the elements of the matrices \( A, B, C, \) and \( D \) are given by

\[
A_{ij} = \frac{W_j}{a^2} \int_0^\infty \left\{ \frac{f(k_i, x)f(k_j, x)}{W_i(x)} + \frac{\eta_0^2}{k_i k_j} \frac{g(k_i, x)g(k_j, x)}{W_0(x)} \right\} \, dx
\]
\[
B_{ij} = \frac{W_j}{a^2} \int_0^\infty \left\{ \frac{f(k_i, x)g(k_j, x)}{W_i(x)} - \frac{\eta_0^2}{k_i k_j} \frac{g(k_i, x)f(k_j, x)}{W_0(x)} \right\} \, dx
\]
\[
C_{ij} = \frac{W_j}{a^2} \int_0^\infty \left\{ \frac{g(k_i, x)f(k_j, x)}{W_i(x)} - \frac{\eta_0^2}{k_i k_j} \frac{f(k_i, x)g(k_j, x)}{W_0(x)} \right\} \, dx
\]
\[
D_{ij} = \frac{W_j}{a^2} \int_0^\infty \left\{ \frac{g(k_i, x)f(k_j, x)}{W_i(x)} + \frac{\eta_0^2}{k_i k_j} \frac{f(k_i, x)f(k_j, x)}{W_0(x)} \right\} \, dx.
\] (27)

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